Lower bounds for the Prékopa-Leindler deficit by some distances modulo translations.

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Abstract

We discuss some refinements of the classical Prékopa-Leindler inequality, which consist in the addition of an extra-term depending on a distance modulo translations. Our results hold true on suitable classes of functions of n variables. They are based upon two different kinds of 1-dimensional refinements: the former is the one obtained by K.M. Ball and K. Böröczky in [4] and involves an L^1-type distance on log-concave functions, the latter is new and involves the transport map onto the Lebesgue measure. Starting from each of these 1-dimensional refinements, we obtain an n-dimensional counterpart by exploiting a generalized version of the Cramér-Wold Theorem.

2010MSC: 52A40, 26D10, 39B62.

Keywords: functional inequalities, Cramér-Wold Theorem, log-concave functions, mass transportation.

1 Introduction

The classical Prékopa-Leindler inequality is the functional form of the Brunn-Minkowski Theorem for the volume functional, see [20, 21, 22, 23]. It states that, given $f, g \in L^1(\mathbb{R}^n; \mathbb{R}_+)$ and $\lambda \in (0,1)$, for any measurable function $h : \mathbb{R}^n \to \mathbb{R}_+$ which satisfies

$$h\big((1-\lambda)x + \lambda y\big) \geq f(x)^{1-\lambda} g(y)^\lambda \qquad \forall x, y \in \mathbb{R}^n \ , \tag{1.1}$$

it holds

$$\int_{\mathbb{R}^n} h \geq \Big(\int_{\mathbb{R}^n} f\Big)^{1-\lambda} \Big(\int_{\mathbb{R}^n} g\Big)^\lambda \ . \tag{1.2}$$

For further information on this inequality and its many applications in different contexts, we refer to [3, 6, 7, 9, 17, 24].

If one introduces the *Prékopa-Leindler deficit* of two given functions $f, g \in L^1(\mathbb{R}^n; \mathbb{R}_+)$ in proportion λ by

$$\gamma_\lambda(f,g) := \inf \Bigg\{ \frac{\int_{\mathbb{R}^n} h}{\Big(\int_{\mathbb{R}^n} f\Big)^{1-\lambda} \Big(\int_{\mathbb{R}^n} g\Big)^\lambda} - 1 \ ; \quad h : \mathbb{R}^n \to \mathbb{R}_+ \text{ measurable verifying (1.1)} \Bigg\} \ , \tag{1.3}$$

inequality (1.2) can be equivalently reformulated as

$$\gamma_\lambda(f,g) \geq 0 \ . \tag{1.4}$$

Notice that, if f, g are Borel functions, then the infimum in (1.3) is attained at

$$h(z) := \sup_{(1-\lambda)x+\lambda y=z} f(x)^{1-\lambda} g(y)^{\lambda} \ ,$$

since such function turns out to be measurable (*cf.* [22, Theorem 3]). However, in the general case when f, g are merely measurable, the above defined function h needs not to be itself measurable (*cf.* [17, Section 10]). Following [9], one might then write the Prékopa-Leindler deficit in terms of the (measurable) function

$$h(z) := \operatorname*{esssup}_{(1-\lambda)x+\lambda y=z} f(x)^{1-\lambda} g(y)^{\lambda} \ ,$$

but in order to avoid the related technicalities, we prefer to deal with the simpler definition (1.3). It is known from S. Dubuc [16] that, assuming that the integrals of f and g are equal 1, if (1.2) holds with equality sign (or equivalently if $\gamma_\lambda(f, g) = 0$), then f agrees almost everywhere with a log-concave function, and there exists $b \in \mathbb{R}^n$ such that

$$f(x) = g(x + b) \ .$$

The purpose of this paper is to provide some improved versions of (1.4), namely to show that, for f, g belonging to suitable classes $\mathcal{A}$ of $L^1(\mathbb{R}^n; \mathbb{R}_+)$, it holds

$$\gamma_\lambda(f, g) \geq \psi_\lambda(d(f, g)) \ , \tag{1.5}$$

where $\psi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a suitable continuous increasing function with $\psi_\lambda(0) = 0$, and d some *distance modulo translations* on $\mathcal{A}$. By this we mean that $d : \mathcal{A} \times \mathcal{A} \to \mathbb{R}_+$ is a symmetric function, satisfying the triangular inequality, and such that

$$d(f, g) = 0 \quad \text{if and only if} \quad f(x) = g(x + b) \text{ for some } b \in \mathbb{R}^n \ .$$

The stability of the Prékopa-Leindler inequality has been recently investigated by K.M. Ball and K. Böröczky [4, 5]. In [4] the authors study the 1-dimensional case, by exploiting the mass transportation of log-concave probability distributions, combined with some of their fine properties holding in dimension 1. In particular, their results entail an inequality of the type (1.5) on the class of compactly supported log-concave probabilities on $\mathbb{R}$, endowed with the following L^1-type distance modulo translations

$$d(f, g) = \inf_{b \in \mathbb{R}} \int_{\mathbb{R}} |f(x) - g(x + b)| \, dx \ . \tag{1.6}$$

Extending this kind of result to higher dimensions is a quite delicate problem. In [5], it is proved that the 1-dimensional refined inequality obtained in [4] continues to hold on the class of *even* log-concave functions on $\mathbb{R}^n$. The restriction to such class of functions is crucial. Indeed, the underlying idea is applying to their (convex and origin-symmetric) level sets the refined version of the Brunn-Minkowski inequality recently proved in [15]: it involves a notion of relative asymmetry for convex bodies which reduces to the Lebesgue measure of their symmetric difference in case they are origin-symmetric. This is the heuristic reason why, after integration, such approach leads to the same kind of L^1-type distance between the initial functions as in dimension $n = 1$, namely

$$d(f, g) = \inf_{b \in \mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - g(x + b)| \, dx \ . \tag{1.7}$$

To the best of our knowledge, this is the unique known stability result for the Prékopa-Leindler inequality in dimension higher than 1. The scope of this paper is to achieve some n-dimensional refinements of the Prékopa-Leindler inequality, whose validity goes out of the class of even functions for some distance of integral type.

Our approach is based on an improved version of the Cramér-Wold Theorem, allowing translations, that we set up in Section 2 (see Theorem 2.2). Though the proof of such extension is quite simple, we believe it may deserve its own interest and have further applications.

Next in Section 3 we use such result to show that an inequality of the kind (1.5) in dimension 1 entails an inequality of the same kind in arbitrary dimension (see Theorem 3.1). We apply this method by taking as a 1-dimensional distance either the one in (1.6) (see Proposition 3.2), or a new one which involves the nondecreasing mass transportation map of a given probability onto the Lebesgue measure on $[0,1]$ (see Proposition 3.5). We point out that these n-dimensional refinements hold on different functional classes, which are quite broad (*cf.* Remarks 3.3 and 3.6). As a drawback, the distance modulo translation that appears in our refinements is more involved than the one in (1.7).

Notwithstanding, the possible interest of our refinements is motivated by their applications in the field of geometric-functional inequalities: as briefly discussed in Section 4, they lead in a natural way to refinements of different kinds of inequalities, such as Brunn-Minkowski type inequalities for variational functionals, as well as isoperimetric-like or log-Sobolev type inequalities on the class of log-concave functions.

2 An improved version of Cramér-Wold Theorem

For every direction $\xi \in S^{n-1}$ (the unit sphere of $\mathbb{R}^n$), let us decompose $\mathbb{R}^n$ as the direct sum of the hyperplane H_ξ through the origin orthogonal to ξ and the linear space spanned by ξ. Thus, for any $x \in \mathbb{R}^n$, we write

$$x = (x', t\xi) \qquad \text{with } x' \in H_\xi \text{ and } t \in \mathbb{R}\ . \tag{2.1}$$

Then, for any given function $f \in L^1(\mathbb{R}^n;\mathbb{R}_+)$, we denote by f_ξ the function in $L^1(\mathbb{R};\mathbb{R}_+)$ defined by

$$f_\xi(t) := \int_{H_\xi} f(x', t\xi)\, d\mathcal{H}^{n-1}(x')\ , \tag{2.2}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure on H_ξ. The Cramér-Wold Theorem [13] states that a Borel probability measure on $\mathbb{R}^n$ is uniquely determined by its 1-dimensional projections. Thus, if f, g are functions in $L^1(\mathbb{R}^n;\mathbb{R}_+)$,

$$\forall \xi \in S^{n-1}\ , \quad f_\xi = g_\xi \quad \text{on } \mathbb{R} \quad \Rightarrow \quad f = g \quad \text{on } \mathbb{R}^n\ .$$

Assume now that the 'slices' f_ξ, g_ξ satisfy a weaker condition, namely they agree up to a translation depending on the direction ξ. What can be inferred on f and g?
Under the assumption that f and g belong to the following space of integrable functions with finite first order momentum

$$L^1_m(\mathbb{R}^n;\mathbb{R}_+) := \Big\{ f \in L^1(\mathbb{R}^n;\mathbb{R}_+)\ :\ xf(x) \in L^1(\mathbb{R}^n;\mathbb{R}^n) \Big\}\,, \tag{2.3}$$

the answer is contained in Theorem 2.2 below.

Remark 2.1. Notice that the finiteness moment condition appearing in (2.3) is obviously satisfied, for instance, by all functions in $L^1(\mathbb{R}^n;\mathbb{R}_+)$ having compact support. We also point out that a similar condition (actually a much stronger form of it, named *Carleman condition*, which requires in particular the finiteness of moments of *all* order), is needed for the validity of the sharp form of the Cramér-Wold Theorem proved in [14].

Theorem 2.2. *Let $f, g \in L^1_m(\mathbb{R}^n;\mathbb{R}_+)$ with nonzero integrals and, for any given $\xi \in S^{n-1}$, let f_ξ, g_ξ be defined as in* (2.2). *Assume that*

$$\forall \xi \in S^{n-1} \ , \ \exists b_\xi \in \mathbb{R} \ : \ f_\xi(t) = g_\xi(t + b_\xi) \quad \text{for a.e. } t \in \mathbb{R} \ . \tag{2.4}$$

Then, setting

$$b := \frac{\int_{\mathbb{R}^n} \big[x g(x) - x f(x) \big] \, dx}{\int_{\mathbb{R}^n} g(x) \, dx} \ , \tag{2.5}$$

it holds

$$b_\xi = b \cdot \xi \quad \forall \xi \in S^{n-1} \qquad \text{and} \qquad f(x) = g(x + b) \quad \text{for a.e. } x \in \mathbb{R}^n \ .$$

Proof. Multiplying by t the identity $f_\xi(t) = g_\xi(t + b_\xi)$ gives

$$\int_{H_\xi} t f(x', t\xi) \, d\mathcal{H}^{n-1}(x') = \int_{H_\xi} t g(x', (t + b_\xi)\xi) \, d\mathcal{H}^{n-1}(x') \ .$$

By exploiting the assumption that the functions f, g belong to $L^1_m(\mathbb{R}^n;\mathbb{R}_+)$ (in particular, the finiteness of their first order momentum), we can now integrate with respect to $t \in \mathbb{R}$. Using the change of variable $s = t + b_\xi$ in the right hand side, we obtain

$$\int_{\mathbb{R}^n} (x \cdot \xi) f(x) \, dx = \int_{\mathbb{R}^n} (x \cdot \xi) g(x) \, dx - b_\xi \int_{\mathbb{R}^n} g(x) \, dx \ .$$

This shows that b_ξ is uniquely determined as the scalar component in direction ξ of the vector $b \in \mathbb{R}^n$ defined in (2.5). Notice that the integral of g at the denominator of (2.5) may be as well replaced by the integral of f, since (2.4) implies in particular $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g$. Notice also that b does not depend on the choice of the coordinate system, since neither the barycenters of f and g, nor their L^1-norm depend on it. Now, if we define

$$\tau_b g(x) := g(x + b) \qquad \forall x \in \mathbb{R}^n \ ,$$

for every $\xi \in S^{n-1}$ it holds

$$\begin{aligned}
(\tau_b g)_\xi(t) \ &= \int_{H_\xi} (\tau_b g)(x', t\xi) \, d\mathcal{H}^{n-1}(x') \\
&= \int_{H_\xi} g(x' + b', (t + b_\xi)\xi) \, d\mathcal{H}^{n-1}(x') \\
&= \int_{H_\xi} g(x', (t + b_\xi)\xi) \, d\mathcal{H}^{n-1}(x') \\
&= g_\xi(t + b_\xi) = f_\xi(t) \ .
\end{aligned}$$

By the Cramér-Wold Theorem, we conclude that $f = \tau_b g$ on $\mathbb{R}^n$. □

As an easy consequence of Theorem 2.2, starting from a distance modulo translations defined on a given class of functions of one variable, one can construct a distance modulo translations in n-dimensions. More precisely, the following statement holds, which paves the way to building refinements of the n-dimensional Prékopa-Leindler inequality, starting from 1-dimensional refinements:

Corollary 2.3. *Let $(\mathcal{A}, d)$ be a subclass of $L^1(\mathbb{R};\mathbb{R}_+)$ endowed with a distance modulo translations, and let $\mathcal{A}_n$ be the subclass of $L^1(\mathbb{R}^n;\mathbb{R}_+)$ defined by*

$$\mathcal{A}_n := \Big\{ f \in L^1_m(\mathbb{R}^n;\mathbb{R}_+) \ : \ f_\xi \in \mathcal{A} \ \ \forall \xi \in S^{n-1} \Big\} \ . \tag{2.6}$$

Then

$$d_n(f,g) := \sup_{\xi \in S^{n-1}} d(f_\xi, g_\xi) \tag{2.7}$$

is a distance modulo translations on $\mathcal{A}_n$.

Proof. Clearly, d_n is a nonnegative and symmetric function on $\mathcal{A}_n$. Moreover, it satisfies the triangular inequality: given functions $f, g, h \in \mathcal{A}_n$, it holds

$$\sup_\xi d(f_\xi, g_\xi) \leq \sup_\xi \Big(d(f_\xi, h_\xi) + d(h_\xi, g_\xi) \Big) \leq \sup_\xi d(f_\xi, h_\xi) + \sup_\xi d(h_\xi, g_\xi) \ .$$

Finally, assume that $d_n(f,g) = 0$. By the definition of d_n, this implies $d(f_\xi, g_\xi) = 0 \ \forall \xi \in S^{n-1}$. Since by assumption d is a distance modulo translation on $\mathcal{A}$, condition (2.4) is satisfied. Then by Theorem 2.2 the functions f and g are translates of each other. □

3 Refinements of Prékopa-Leindler inequality

By following the proof of the n-dimensional Prékopa-Leindler inequality by induction on n (*cf.* [18, Theorem 4.2]), and exploiting Corollary 2.3, we obtain:

Theorem 3.1. *Let $(\mathcal{A}, d)$ be a subclass of $L^1(\mathbb{R};\mathbb{R}_+)$ endowed with a distance modulo translations, and let $(\mathcal{A}_n, d_n)$ be the corresponding subclass of $L^1(\mathbb{R}^n;\mathbb{R}_+)$ as in Corollary 2.3. Assume that, for some continuous increasing function $\psi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi_\lambda(0) = 0$, it holds*

$$\gamma_\lambda(F,G) \geq \psi_\lambda\big(d(F,G)\big) \qquad \forall F, G \in \mathcal{A} \ . \tag{3.1}$$

Then it holds

$$\gamma_\lambda(f,g) \geq \psi_\lambda\big(d_n(f,g)\big) \qquad \forall f, g \in \mathcal{A}_n \ . \tag{3.2}$$

Proof. Let $h : \mathbb{R}^n \to \mathbb{R}_+$ be a measurable function satisfying (1.1), and select an arbitrary direction $\xi \in S^{n-1}$; accordingly, we decompose any $x \in \mathbb{R}^n$ as in (2.1). Then, we fix $a, b \in \mathbb{R}$ and we set $c := (1-\lambda)a + \lambda b$. We observe that, by the choice of h and c, the functions

$$x' \mapsto f(x', a\xi), \quad g(x', b\xi), \quad h(x', c\xi)$$

satisfy the assumption of the Prékopa-Leindler inequality (namely the inequality (1.1)) on H_ξ. Therefore, if f_ξ, g_ξ and h_ξ are the functions of one real variable defined according to (2.2), it holds

$$h_\xi(c) \geq \big(f_\xi(a)\big)^{1-\lambda} \big(g_\xi(b)\big)^\lambda \ .$$

By the arbitrariness of a and b in $\mathbb{R}$, the definition of c, and assumption (3.1), it follows

$$\int_{\mathbb{R}} h_\xi \geq \Big(\int_{\mathbb{R}} f_\xi \Big)^{1-\lambda} \Big(\int_{\mathbb{R}} g_\xi \Big)^{\lambda} \Big(1 + \psi_\lambda \big(d(f_\xi, g_\xi) \big) \Big) \ ,$$

that is

$$\int_{\mathbb{R}^n} h \geq \Big(\int_{\mathbb{R}^n} f \Big)^{1-\lambda} \Big(\int_{\mathbb{R}^n} g \Big)^{\lambda} \Big(1 + \psi_\lambda \big(d(f_\xi, g_\xi) \big) \Big) \ .$$

Finally, we pass to the supremum with respect to $\xi \in S^{n-1}$ at the right hand side. Taking into account that ψ_λ is continuous and increasing, (3.2) follows. □

The remaining of this section is devoted to exemplify the applications of Theorem 3.1, by taking as $(\mathcal{A}, d)$ two different classes of functions of one variable, endowed with two different kinds of distances modulo translations. Firstly, we consider the 1-dimensional refinement obtained by K.M. Ball and K. Böröczky in [4] on the class of log-concave functions and we obtain:

Proposition 3.2. *Set*

$$\mathcal{A} := \Big\{ F \in L^1(\mathbb{R}; \mathbb{R}_+) \ : \ \int_{\mathbb{R}} F = 1 \ , \ F \text{ is log-concave} \Big\} \ , \tag{3.3}$$

$$d(F, G) := \inf_{b \in \mathbb{R}} \int_{\mathbb{R}} |F(x) - G(x+b)| \, dx \ , \tag{3.4}$$

and define $\mathcal{A}_n$ and d_n according to Corollary 2.3. We have:

(i) d and d_n are distances modulo translations respectively on $\mathcal{A}$ and $\mathcal{A}_n$;

(ii) inequalities (3.1) *and* (3.2) *hold by taking ψ_λ as the inverse function on $\mathbb{R}^+$ of*

$$\varphi_\lambda(t) := c(\lambda) \, t^{1/3} |\log t|^{4/3} (1+t) \ ,$$

being $c = c(\lambda)$ a suitable positive constant.

Remark 3.3. As noticed in [4], it is unclear whether the log-concavity assumption appearing in the class $\mathcal{A}$ defined by (3.3) is actually necessary for the validity of (3.1). Let us also point out that the corresponding class $\mathcal{A}_n$ where (3.2) holds true, turns out to contain all log-concave functions f in $L^1(\mathbb{R}^n; \mathbb{R}_+)$ such that $\int_{\mathbb{R}^n} f = 1$ (see Corollary 3.5 in [9] or Theorem 11.2 of the extended version [18] of the paper [17], which reduce the problem to $n = 1$, where the assertion is trivial).

Proof. (i) By definition, d is a nonnegative symmetric function on $\mathcal{A}$. Moreover, if $d(F, G) = 0$, there exists a sequence $\{b_k\} \subset \mathbb{R}$ such that

$$\lim_k \int_{\mathbb{R}} |F(x) - G(x + b_k)| \, dx = 0 \ . \tag{3.5}$$

We observe that such a sequence $\{b_k\}$ is necessarily bounded. Indeed, if (up to a subsequence) it holds $|b_k| \to +\infty$, since G is log-concave and integrable it holds $G(x + b_k) \to 0$ for a.e. $x \in \mathbb{R}$, and by Fatou's Theorem we get $\int_{\mathbb{R}} |F(x)| = 0$, contradiction. Hence (again up to a subsequence), we have $b_k \to b$ for some $b \in \mathbb{R}$. Taking into account that the functions in $\mathcal{A}$ are log-concave, and in particular continuous on the interior of their support, it holds

$$\lim_k G(x + b_k) = G(x + b) \qquad \text{for a.e. } x \in \mathbb{R} \ . \tag{3.6}$$

Then, by (3.5), (3.6) and Fatou's Theorem, we have

$$\int_{\mathbb{R}} |F(x) - G(x+b)|\, dx = 0 \ ,$$

whence F and G are translates of each other. Finally, given functions $F, G, H \in \mathcal{A}$, for every $\varepsilon > 0$ there exist b_1 and b_2 such that

$$d(F,H) \geq \int_{\mathbb{R}} |F(x) - H(x+b_1)|\, dx - \varepsilon \qquad \text{and} \qquad d(H,G) \geq \int_{\mathbb{R}} |H(x) - G(x+b_2)|\, dx - \varepsilon \ .$$

Then

$$\begin{aligned} d(F,G) &\leq \int_{\mathbb{R}} |F(x) - G(x+b_1+b_2)|\, dx \\ &\leq \int_{\mathbb{R}} |F(x-b_1) - H(x)|\, dx + \int_{\mathbb{R}} |H(x) - G(x+b_2)|\, dx \\ &\leq d(F,H) + d(H,G) + 2\varepsilon \ , \end{aligned}$$

and the triangular inequality follows by letting ε tend to zero.

(ii) Inequality (3.1) follows directly from [4, Theorem 1.2] (see also Remarks 1.3 and 1.6 in [4], and the Remark just after Theorem 1.2 in [5]). Inequality (3.2) is an immediate consequence of inequality (3.1) and Theorem 3.1. □

Remark 3.4. The distance constructed in Proposition 3.2 is clearly weaker than the L^1 distance (1.7). Indeed,

$$\begin{aligned} d_n(f,g) &= \sup_{\xi \in S^{n-1}} \inf_{b \in \mathbb{R}} \int_{\mathbb{R}} |f_\xi(t) - g_\xi(t+b)| dt, \\ &= \sup_{\xi \in S^{n-1}} \inf_{b \in \mathbb{R}} \int_{\mathbb{R}} \Big| \int_{H_\xi} f(x', t\xi) - g(x', (t+b)\xi)\, d\mathcal{H}^{n-1}(x') \Big| dt. \end{aligned}$$

Denoting b_* the minimizer in (1.7), and $b'_*, b_{*,\xi}$ its projections on H_ξ and $\xi\mathbb{R}$, respectively, we have

$$\begin{aligned} d_n(f,g) &\leq \sup_{\xi \in S^{n-1}} \int_{\mathbb{R}} \Big| \int_{H_\xi} f(x', t\xi) - g(x' + b'_*, (t + b_{*\xi})\xi)\, d\mathcal{H}^{n-1}(x') \Big| dt \\ &\leq \sup_{\xi \in S^{n-1}} \inf_{c \in \mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - g(x+c)| dx = \inf_{c \in \mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - g(x+c)| dx. \end{aligned}$$

Next we propose a different kind of 1-dimensional refinement, which involves the transportation map of an absolutely continuous probability onto the Lebesgue measure on $[0,1]$.
With any given function $F \in L^1(\mathbb{R}; \mathbb{R}_+)$ with $\int_{\mathbb{R}} F = 1$, we associate the map $u_F : [0,1] \to \mathbb{R}$ such that $u_F(t)$ is the smallest number satisfying

$$\int_{-\infty}^{u_F(t)} F(x)\, dx = t \ . \tag{3.7}$$

Below, $F(u_F)$ stands for the composition $F \circ u_F$, $\mathcal{L}^1$ for the Lebesgue measure on $\mathbb{R}$, and $AC(0,1)$ for the class of absolutely continuous functions on $(0,1)$.

Proposition 3.5. *Set*

$$\mathcal{A} := \left\{ F \in L^1(\mathbb{R};\mathbb{R}_+) \ : \ \int_{\mathbb{R}} F = 1 \ , \ u_F \in AC(0,1) \right\} , \tag{3.8}$$

$$d(F,G) := \mathcal{L}^1\big(\{F(u_F) \neq G(u_G)\}\big) - \int_{\{F(u_F)<G(u_G)\}} \frac{F(u_F)}{G(u_G)} - \int_{\{G(u_G)<F(u_F)\}} \frac{G(u_G)}{F(u_F)} \ , \tag{3.9}$$

and define $\mathcal{A}_n$ *and* d_n *according to Corollary 2.3. We have:*

(i) d *and* d_n *are distances modulo translations respectively on* $\mathcal{A}$ *and* $\mathcal{A}_n$*;*

(ii) inequalities (3.1) *and* (3.2) *hold by taking*

$$\psi_\lambda(t) := c(\lambda)\, t^2 \ ,$$

being $c(\lambda)$ *a positive constant depending on* λ*.*

Remark 3.6. (i) Removing the condition $u_F \in AC(0,1)$ in (3.8), the function d in (3.9) would be no longer a distance modulo translations. Indeed, for given positive numbers $a < b$, let

$$F(x) := \chi_{[0,a]\cup[b,b+1-a]} \qquad \text{and} \qquad G(x) := \chi_{[0,1]} \ .$$

It is easy to check that

$$u_F(t) = \begin{cases} t & \text{if } t \in [0,a] \\ t+b-a & \text{if } t \in (a,1] \ , \end{cases}$$

which in particular shows that $u_F \notin AC(0,1)$. Moreover, it holds $F(u_F) = G(u_G) = 1$ on $[0,1]$, so that $d(F,G) = 0$. But clearly F and G do not agree up to a translation of their variable.

(ii) The condition $u_F \in AC(0,1)$ in (3.8) is satisfied for instance for all continuous functions F such that $\mathrm{spt}(F) = [a,b]$ and the set $\{x \in [a,b] \ : \ F(x) = 0\}$ is empty or given by a finite number of points. Indeed in this case the map $t \mapsto \int_a^t F(s)\,ds$ is of class C^1 and strictly increasing from $[a,b]$ onto $[0,1]$, and u_F is precisely its inverse.

Consequently, the corresponding class $\mathcal{A}_n$ where (3.2) holds true, turns out to contain for instance all continuous functions $f \in L^1(\mathbb{R}^n;\mathbb{R}_+)$ with $\int_{\mathbb{R}^n} f = 1$ such that $\mathrm{spt}(f)$ is a compact set and f is strictly positive on its interior.

(iii) We point out that the class (3.8) contains functions which are not log-concave and hence do not lie in the class (3.3). For instance, one can check that the function

$$F(x) := (\alpha+1)x^\alpha \chi_{[0,1]}(x)$$

belongs to the class (3.8) as soon as $\alpha > -1$, whereas it is log-concave only for $\alpha > 0$.

Proof of Proposition 3.5. (i) Clearly d is a nonnegative symmetric function on $\mathcal{A}$. Moreover, if $d(F,G) = 0$, from the definition of d it follows

$$F(u_F) = G(u_G) \qquad \text{a.e. on } [0,1] \ . \tag{3.10}$$

Since by differentiating (3.7) one sees that $F(u_F(t))u_F'(t) = 1$ a.e. on $[0,1]$, condition (3.10) implies

$$u_F'(t) = u_G'(t) \qquad \text{a.e. on } [0,1] \ . \tag{3.11}$$

In turn, since u_F and u_G are absolutely continuous maps, (3.11) yields the existence of some $b \in \mathbb{R}$ such that $u_F(t) = u_G(t) + b$, and hence by (3.10) the functions F and G are translates of each other. In order to check the triangular inequality, it is convenient to rewrite $d(F,G)$ as

$$d(F,G) = \int_{\{F(u_F)<G(u_G)\}} \Big[1 - \frac{F(u_F)}{G(u_G)}\Big] + \int_{\{G(u_G)<F(u_F)\}} \Big[1 - \frac{G(u_G)}{F(u_F)}\Big] \ .$$

Then, given functions $F, G, H \in \mathcal{A}$, we observe that

$$d(F,G) - d(F,H) - d(H,G) = \sum_{i=1}^{3} \int_{\omega_i} \Phi_i + \sum_{i=1}^{3} \int_{\tilde{\omega}_i} \widetilde{\Phi}_i \ , \tag{3.12}$$

where

$$\omega_1 := \big\{x \in \mathbb{R} \ : \ H(u_H) \le F(u_F) \le G(u_G)\big\} \ , \quad \Phi_1 := \Big[1 - \frac{F(u_F)}{G(u_G)}\Big] - \Big[1 - \frac{H(u_H)}{F(u_F)}\Big] - \Big[1 - \frac{H(u_H)}{G(u_G)}\Big]$$

$$\omega_2 := \big\{x \in \mathbb{R} \ : \ F(u_F) \le H(u_H) \le G(u_G)\big\} \ , \quad \Phi_2 := \Big[1 - \frac{F(u_F)}{G(u_G)}\Big] - \Big[1 - \frac{F(u_F)}{H(u_H)}\Big] - \Big[1 - \frac{H(u_H)}{G(u_G)}\Big]$$

$$\omega_3 := \big\{x \in \mathbb{R} \ : \ F(u_F) \le G(u_G) \le H(u_H)\big\} \ , \quad \Phi_3 := \Big[1 - \frac{F(u_F)}{G(u_G)}\Big] - \Big[1 - \frac{F(u_F)}{H(u_H)}\Big] - \Big[1 - \frac{G(u_G)}{H(u_H)}\Big] \ ,$$

and the sets $\tilde{\omega}_i$ and the functions $\widetilde{\Phi}_i$ are defined in the analogous way simply exchanging the roles of F and G. Some straightforward algebraic computations show that for every $i = 1, 2, 3$, the function Φ_i (resp. $\widetilde{\Phi}_i$) is nonpositive on the set ω_i (resp $\tilde{\omega}_i$) and therefore the triangular inequality follows from (3.12).

(ii) In order to prove inequality (3.1), we proceed along the line of the second proof of the 1-dimensional Prékopa-Leindler inequality given in [18, Section 4]. Let $H : \mathbb{R} \to \mathbb{R}_+$ be a measurable function satisfying

$$H((1-\lambda)x + \lambda y) \ge F(x)^{1-\lambda} G(y)^{\lambda} \qquad \forall x, y \in \mathbb{R} \ ,$$

and, for $t \in \mathbb{R}$, let

$$w(t) := (1-\lambda) u_F(t) + \lambda u_G(t) \ .$$

Setting for brevity $f := F \circ u_F$ and $g := G \circ u_G$, one has:

$$\begin{aligned} \int_{\mathbb{R}} H \ &\ge \int_0^1 H(w(t)) w'(t) \, dt \\ &\ge \int_0^1 f^{1-\lambda} g^{\lambda} \ \big[(1-\lambda) u_F' + \lambda u_G'\big] \, dt \\ &= \int_0^1 f^{1-\lambda} g^{\lambda} \ \big[(1-\lambda) \frac{1}{f} + \lambda \frac{1}{g}\big] \, dt \ . \end{aligned} \tag{3.13}$$

Now we recall that, given positive numbers x, y, and $\lambda \in (0,1)$, setting $\mathcal{A} := (1-\lambda)x + \lambda y$ and $\mathcal{G} := x^{1-\lambda} y^{\lambda}$, the following version of the arithmetic-geometric inequality holds [2]:

$$\mathcal{A} - \mathcal{G} \ge \frac{1}{2 \max\{x, y\}} [(1-\lambda)(x - \mathcal{G})^2 + \lambda (y - \mathcal{G})^2] \ . \tag{3.14}$$

By (3.13) and (3.14), it holds

$$\begin{aligned}\gamma_\lambda(F,G) &= \int_{\mathbb{R}} H - 1 \geq \frac{1}{2}\int_0^1 \frac{f^{1-\lambda}g^\lambda}{\max\{\frac{1}{f},\frac{1}{g}\}}\Big[(1-\lambda)\Big(\frac{1}{f}-\frac{1}{f^{1-\lambda}g^\lambda}\Big)^2+\lambda\Big(\frac{1}{g}-\frac{1}{f^{1-\lambda}g^\lambda}\Big)^2\Big]\,dt\\ &= \frac{1}{2}\int_0^1 \frac{f^{1-\lambda}g^\lambda}{\max\{\frac{1}{f},\frac{1}{g}\}}\Big[\frac{(1-\lambda)}{f^{2(1-\lambda)}}\Big(\frac{1}{f^\lambda}-\frac{1}{g^\lambda}\Big)^2+\frac{\lambda}{g^{2\lambda}}\Big(\frac{1}{f^{1-\lambda}}-\frac{1}{g^{1-\lambda}}\Big)^2\Big]\,dt\ .\end{aligned}$$

Then, by using the elementary inequalities

$$|x^\alpha - y^\alpha| \geq \alpha\max\{x,y\}^{\alpha-1}|x-y| \qquad \forall x,y>0\ ,\ \alpha\in(0,1)\ ,$$

and

$$x+y \geq x^\lambda y^{1-\lambda} \qquad \forall x,y>0\ ,\ \lambda\in(0,1)\ ,$$

we get

$$\begin{aligned}\gamma_\lambda(F,G) &\geq \frac{1}{2}\int_0^1 \frac{f^{1-\lambda}g^\lambda}{\max\{\frac{1}{f},\frac{1}{g}\}}\Big|\frac{1}{f}-\frac{1}{g}\Big|^2\Big[\frac{\lambda^2(1-\lambda)}{f^{2(1-\lambda)}}\max\Big\{\frac{1}{f},\frac{1}{g}\Big\}^{2(\lambda-1)}+\frac{\lambda(1-\lambda)^2}{g^{2\lambda}}\max\Big\{\frac{1}{f},\frac{1}{g}\Big\}^{-2\lambda}\Big]\,dt\\ &\geq \int_0^1 \frac{1}{\max\{\frac{1}{f},\frac{1}{g}\}^2}\Big|\frac{1}{f}-\frac{1}{g}\Big|^2 f^{(1-\lambda)(1-2\lambda)}g^{\lambda-2\lambda(1-\lambda)}\max\Big\{\frac{1}{f},\frac{1}{g}\Big\}^{4\lambda(\lambda-1)+1}\,dt\\ &\geq c(\lambda)\int_0^1 \frac{1}{\max\{\frac{1}{f},\frac{1}{g}\}^2}\Big|\frac{1}{f}-\frac{1}{g}\Big|^2\,dt\ .\end{aligned}$$

Finally we apply the Hölder inequality to obtain

$$\gamma_\lambda(F,G) \geq c(\lambda)\Big[\int_0^1 \min\{f,g\}\Big|\frac{1}{f}-\frac{1}{g}\Big|\,dt\Big]^2$$

This concludes the proof of inequality (3.1) by noticing that

$$\begin{aligned}\int_0^1 \min\{f,g\}\Big|\frac{1}{f}-\frac{1}{g}\Big|\,dt &= \int_{\{\frac{1}{g}<\frac{1}{f}\}}\Big(\frac{1}{f}-\frac{1}{g}\Big)f\,dt+\int_{\{\frac{1}{f}<\frac{1}{g}\}}\Big(\frac{1}{g}-\frac{1}{f}\Big)g\,dt\\ &= \int_{\{f<g\}}\Big(1-\frac{f}{g}\Big)\,dt+\int_{\{g<f\}}\Big(1-\frac{g}{f}\Big)\,dt\\ &= \mathcal{L}^1\big(\{f\neq g\}\big)-\int_{\{f<g\}}\frac{f}{g}\,dt-\int_{\{g<f\}}\frac{g}{f}\,dt\ .\end{aligned}$$

Inequality (3.2) is an immediate consequence of inequality (3.1) and Theorem 3.1.

□

4 Applications

4.1 Refined Brunn-Minkowski type inequalities for variational functionals

Among its many applications, the Prékopa-Leindler has been used in the literature in order to obtain Brunn-Minkowski type inequalities for variational functionals. Our results yield refined

versions of such inequalities, the extra-term involving a certain distance between the solutions to the corresponding problems (provided such solutions fall into one of the appropriate classes of functions).
As an example, let us detail a refined Brunn-Minkowski inequality for the torsional rigidity.
Recall that the torsional rigidity of an open bounded domain Ω of $\mathbb{R}^n$ is defined as

$$\tau(\Omega) = \int_\Omega |\nabla u_\Omega|^2 \, dx = \int_\Omega u_\Omega \, dx \ ,$$

being u_Ω the unique solution in $H^1_0(\Omega)$ to the Dirichlet problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{4.1}$$

In dimension n, the shape functional $\tau(\cdot)$ is easily checked to be homogeneous of degree $n+2$ under domain dilations. Moreover, on the class $\mathcal{K}^n$ of convex bodies (convex compact sets with nonempty interior in $\mathbb{R}^n$), it is known to satisfy the following Brunn-Minkowski type inequality [8]

$$\tau((1-\lambda)K_0 + \lambda K_1)^{1/(n+2)} \geq (1-\lambda)\tau(K_0)^{1/(n+2)} + \lambda\tau(K_1)^{1/(n+2)} \qquad \forall K_0, K_1 \in \mathcal{K}^n \ , \ \lambda \in (0,1) \ ,$$

and the inequality is an equality if and only if K_0 and K_1 are homothetic.
As a consequence of Proposition 3.2, such an inequality holds in the following refined form:

Proposition 4.1. *Let $K_0, K_1 \in \mathcal{K}^n$, and denote by u_0, u_1 the corresponding solutions to problem* (4.1)*. For every $\lambda \in (0,1)$, there holds*

$$\tau((1-\lambda)K_0 + \lambda K_1)^{1/(n+2)} \geq \Big[(1-\lambda)\tau(K_0)^{1/(n+2)} + \lambda\tau(K_1)^{1/(n+2)}\Big] \cdot \Big[1 + d_n\Big(\frac{u_0}{\tau(K_0)}, \frac{u_1}{\tau(K_1)}\Big)\Big] \ ,$$

being d_n the distance modulo translations defined in (2.7)*, with d as in* (3.4)*.*

Proof. We follow the approach adopted in the proof of Theorem 11 in [11]. We set $K_\lambda := (1-\lambda)K_0 + \lambda K_1$, and we denote by u_λ the solution to problem (4.1) on K_λ. After extending them to 0 respectively outside K_0, K_1, K_λ, we may consider the functions u_0, u_1, u_λ as defined on the whole space $\mathbb{R}^n$. By [11, Theorem 20] and the arithmetic-geometric inequality, it holds

$$u_\lambda((1-\lambda)x + \lambda y) \geq u_0(x)^{1-\lambda} u_1(y)^\lambda \qquad \forall x, y \in \mathbb{R}^n \ . \tag{4.2}$$

Moreover, taking the class $\mathcal{A}$ as in (3.3) and denoting by $\mathcal{A}_n$ the corresponding class given by (2.6), it holds

$$\frac{u_i}{\tau(K_i)} \in \mathcal{A}_n \qquad i = 0, 1 \ . \tag{4.3}$$

Indeed, it is known that $\sqrt{u_i}$ are concave (see [19] and also [1, 10]). Hence in particular u_i are log-concave, which implies (4.3) recalling Remark 3.3. By (4.2) and (4.3), Proposition 3.2 yields

$$\tau(K_\lambda) \geq \tau(K_0)^{1-\lambda}\tau(K_1)^\lambda \Big[1 + d_n\Big(\frac{u_0}{\tau(K_0)}, \frac{u_1}{\tau(K_1)}\Big)\Big] \ . \tag{4.4}$$

The proposition follows by applying (4.4) to

$$K_0' := \frac{K_0}{\tau(K_0)^{1/(n+2)}} \ , \qquad K_1' := \frac{K_1}{\tau(K_1)^{1/(n+2)}} \ , \qquad \lambda' := \frac{\lambda\tau(K_1)^{1/(n+2)}}{(1-\lambda)\tau(K_0)^{1/(n+2)} + \lambda\tau(K_1)^{1/(n+2)}} \ .$$

□

4.2 Refined Minkowski first inequality for log-concave functions

A functional form of Minkowski first inequality has been recently obtained in [12], settled in a suitable class of log-concave functions on $\mathbb{R}^n$. Let us show how such an inequality can be improved, by exploiting the Prékopa-Leindler inequality in refined form. We need to introduce some notation from [12]. Set

$$\mathcal{L} := \Big\{ u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \ \Big| \ u \text{ proper, convex, } \lim_{\|x\| \to +\infty} u(x) = +\infty \ . \Big\}$$

For any log-concave function of the form $f = e^{-u}$, with $u \in \mathcal{L}$, consider the total mass functional

$$J(f) = \int_{\mathbb{R}^n} f \, dx \in [0, +\infty) \ .$$

Moreover, if $g = e^{-v}$ for some $v \in \mathcal{L}$, and $\alpha, \beta \in \mathbb{R}_+$, set

$$(\alpha \cdot f \ \oplus \ \beta \cdot g)(x) := \sup_{y \in \mathbb{R}^n} \ f\left(\frac{x-y}{\alpha}\right)^{\alpha} g\left(\frac{y}{\beta}\right)^{\beta} \ .$$

In this algebraic structure, according to [12, Theorem 3.6], the following derivative turns out to exist as soon as $J(f) > 0$:

$$\delta J(f, g) := \lim_{t \to 0^+} \frac{J(f \oplus t \cdot g) - J(f)}{t}$$

Then, the functional form of Minkowski first inequality for log-concave functions established in [12, Theorem 5.1] reads

$$\delta J(f, g) \geq J(f)\big[\log J(g) + n\big] + \int_{\mathbb{R}^n} f \log f \, dx - J(f) \log J(f) \ , \tag{4.5}$$

and the inequality is an equality if and only if $g(x) = f(x + b)$ for some $b \in \mathbb{R}^n$.
Let us show how a lower bound on the Prékopa-Leindler deficit of f and g produces an extra-term in the right hand side of (4.5):

Proposition 4.2. *Let $f = e^{-u}$ and $g = e^{-v}$, with $u, v \in \mathcal{L}$. Assume that the Prékopa-Leindler deficit of f and g satisfies a lower bound of the kind*

$$\gamma_\lambda(f, g) \geq \psi_\lambda(d(f, g)) \ ,$$

where d is some distance modulo translations and $\psi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous increasing function with $\psi_\lambda(0) = 0$. Assume in addition that the map $\lambda \mapsto \psi_\lambda(\cdot)$ vanishes and is differentiable at $\lambda = 0^+$. Then (4.5) can be improved into

$$\delta J(f, g) \geq J(f)\big[\log J(g) + n\big] + \int_{\mathbb{R}^n} f \log f \, dx - J(f) \log J(f) + \frac{d}{d\lambda} \psi_\lambda(d(f, g))_{|\lambda=0} \ . \tag{4.6}$$

Proof. By the assumption $\gamma_\lambda(f, g) \geq \psi_\lambda(d(f, g))$, it holds

$$J[(1-\lambda) \cdot f \oplus \lambda \cdot g] \geq J(f)^{1-\lambda} J(g)^{\lambda} \Big[1 + \psi_\lambda(d(f, g))\Big] \qquad \forall \lambda \in [0, 1] \ .$$

Hence, the function $\psi(\lambda) := \log\big(J((1-\lambda) \cdot f \oplus \lambda \cdot g)\big)$ satisfies

$$\psi(\lambda) \geq \psi(0) + \lambda\big[\psi(1) - \psi(0)\big] + \log[1 + \psi_\lambda(d(f, g))] \qquad \forall t \in [0, 1] \ .$$

As a consequence, the (right) derivative of ψ at $\lambda = 0$ satisfies

$$\psi'(0) \geq \big[\psi(1) - \psi(0)\big] + \frac{d}{d\lambda}\psi_\lambda(d(f,g))_{|\lambda=0} \ . \tag{4.7}$$

By Lemma 5.4 in [12], we have

$$\psi'(0) = \frac{\delta J(f,g) - \delta J(f,f)}{J(f)} \ .$$

Therefore (4.7) can be rewritten as

$$\frac{\delta J(f,g) - \delta J(f,f)}{J(f)} \geq \log\Big(\frac{J(g)}{J(f)}\Big) + \frac{d}{d\lambda}\psi_\lambda(d(f,g))_{|\lambda=0} \ .$$

Inserting into the above inequality the expression of $\delta J(f,f)$ given by [12, Proposition 3.11], (4.6) is proved. □

We point out that Proposition 4.2 may be helpful to obtain refined formulations of other functional inequalities such as the functional isoperimetric inequality (*cf.* [12, Proposition 6.1]) or log-Sobolev type inequalities with respect to log-concave probability densities (*cf.* [12, Proposition 6.2]). In fact, the quite abstract formulation of inequality (4.6) becomes more explicit when dealing with suitable subclasses of log-concave functions where the representation formulae obtained in [12] for $\delta J(f,g)$ hold true. On the other hand, it is clear that such strategy may be successfull only if the extra-term in (4.6) is nonzero, namely only if

$$\frac{d}{d\lambda}\psi_\lambda(d(f,g))_{|\lambda=0} \neq 0 \ . \tag{4.8}$$

Let us notice that such condition is satisfied when considering the lower bound for the Prékopa-Leindler deficit provided by Proposition 3.5. Indeed, by direct inspection of the proof, one can easily check that in that case there holds

$$\psi_\lambda(t) = c(\lambda)t^2 \ , \qquad \text{with} \ \ c(\lambda) = \lambda^{1+\lambda}(1-\lambda)^{2-\lambda} \ ,$$

so that $c(0) = 0$ and $c'(0) = 1$.
The situation is more delicate when considering the lower bound for the Prékopa-Leindler deficit provided by Proposition 3.2. Indeed the explicit dependence on λ of the constant $c(\lambda)$ appearing therein (which comes from the work [4] of Ball and Böröczky) is by now not available, so that at this stage we are not able to verify the validity of (4.8), and some further investigation is needed in this respect.

Acknowledgments. This work was realized thanks to the financial support of the French Agence Nationale de la Recherche, as part of the project ANR-09-BLAN-0037 "Geometric analysis of optimal shapes", and of the Italian Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni, as part of the project "Problemi di ottimizzazione e disuguaglianze geometrico-funzionali".
We thank the referee for his/her careful reading of the paper, which allowed us to precise some points in the results.

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