Minimization of eigenvalues by free boundary - free discontinuity methods

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I this talk I reported on new methods based on free boundary - free discontinuity techniques that can be used in order to obtain qualitative information on the optimal domains minimizing a shape functional. In particular, I considered functionals depending on the spectrum of the Laplace operator with Dirichlet or Robin boundary condition.

Dirichlet boundary conditions. Let $F : \mathbb{R}^k \to \mathbb{R}$ be a lower semicontinuous function, non-decreasing in each variable. For every quasi-open set $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$) we denote

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq ... \leq \lambda_k(\Omega)$$

the first k eigenvalues of the Laplacian with Dirichlet boundary conditions. Concerning the definition of quasi-open sets and of the Laplace operator on quasi-open sets, we refer to [3, Chapters 4, 5]. Denoting by $|\Omega|$ the measure of Ω, for some $m > 0$ we consider the following problems:

$$(P_1) \qquad \min_{\Omega \text{ quasi-open}, |\Omega| = m} F(\lambda_1(\Omega), ..., \lambda_k(\Omega));$$

$$(P_2) \qquad \min_{\Omega \text{ quasi-open}} F(\lambda_1(\Omega), ..., \lambda_k(\Omega)) + |\Omega|.$$

Below is a compilation, presented in a unified framework, of two recent existence-qualitative results obtained independently and using different methods by Mazzoleni and Pratelli [8] and D. B. [2].

a) Under the hypotheses above, problems (P_1) and (P_2) have at least one solution Ω, which is a bounded set.
b) If F is moreover locally Lipschitz, then every solution Ω of (P_2) is a bounded set with finite perimeter.
c) If in addition F is locally bi-Lipschitz in at least one variable, then every solution Ω of (P_1) is a bounded set with finite perimeter.

The existence result for (P_1) and (P_2) under the additional constraint that the competing domains are contained in a prescribed bounded open set was proved by Buttazzo and Dal Maso in [6]. Point a) is proved in [8] and is based on a geometric argument allowing to cut small (unbounded) regions of a set while keeping control on the spectrum. Points b) and c) are a consequence of the analysis of the shape subsolutions by free boundary methods (see [2]) and can be complemented with more information related to the inner density of the optimal shapes. In all cases, the diameter of the optimal sets are controlled.

Robin boundary conditions. Let $\beta > 0$ be fixed. For every bounded set with Lipschitz boundary $\Omega \subseteq \mathbb{R}^N$ we consider the Robin-torsional rigidity defined by

$P(\Omega) = \int_\Omega u dx$, where u solves the problem

$$\begin{cases} -\Delta u = & 1 \text{ in } \Omega \\ \frac{\partial u}{\partial n} + \beta u = & 0 \ \partial\Omega \end{cases}$$

In a joint work with A. Giacomini (in progress) we prove that

$$P(B) \geq P(\Omega),$$

where B is the ball of the same measure as Ω. Moreover, equality holds if and only if Ω is the ball.

The method is based on a free discontinuity approach, via the analysis of the relaxed free discontinuity problem

$$\min_{u^2 \in SBV(\mathbb{R}^N), |\{u>0\}=m|} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\beta}{2}\int_{J_u} (|u^+|^2 + |u^-|^2) d\mathcal{H}^{N-1} - \int_{\mathbb{R}^N} u dx,$$

for which we prove existence of a solution and Ahlfors regularity of the jump set. Of crucial importance in the proof of the Ahlfors regularity is the monotonicity lemma obtained in a joint work with S. Luckhaus [4], as well as a regularity result for free discontinuity problems with Robin boundary conditions obtained in the same paper, asserting the existence of a positive constant $\alpha > 0$ such that the solution of the free discontinuity problem above satisfies

$$u(x) \geq \alpha \quad \text{a.e.} \quad x \in \{u > 0\}.$$

This results would be a consequence of the Hopf maximum principle if the boundary of the positivity region were smooth.

Consequently, one proves that the minimizer of the relaxed free discontinuity problem above consists on a couple: an open set with finite perimeter Ω and the Robin torsion function on Ω extended by zero on its complement. Radial symmetry of the set Ω and its torsion function is obtained by a cut and reflect argument.

This method can be extended to semi-linear eigenvalue Robin problems

$$\min_{|\Omega|=m} \min_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} |u|^2 d\mathcal{H}^{N-1}}{\left(\int_\Omega |u|^q dx\right)^{\frac{2}{q}}},$$

for $1 \leq q < \frac{2N}{N-2}$. The first Robin eigenvalue of the Laplacian corresponds to the case $q = 2$ and the torsional rigidity to $q = 1$. In the literature, only the case $q = 2$ was known (Bossel [1] in $\mathbb{R}^2$, Daners [7] in $\mathbb{R}^N$ within a Lipschitz setting and B.-Giacomini [5] for the general SBV framework). The proof of the case $q = 2$ uses a key idea of Bossel related to the notion of extremal lengths which does not seem (easily) extendable to other values of q.

References

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