BLASCHKE-SANTALÓ AND MAHLER INEQUALITIES FOR THE FIRST EIGENVALUE OF THE DIRICHLET LAPLACIAN

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Abstract. For K belonging to the class of convex bodies in $\mathbb{R}^n$, we consider the λ_1-product functional, defined by $\lambda_1(K)\lambda_1(K^o)$, where K^o is the polar body of K, and $\lambda_1(\cdot)$ is the first Dirichlet eigenvalue of the Dirichlet Laplacian. As a counterpart of the classical Blaschke-Santaló inequality for the volume product, we prove that the λ_1-product is minimized by balls. Much more challenging is the problem of maximizing the λ_1-product modulo invertible linear transformations, which is the analogue of the famous Mahler conjecture for the volume product in Convex Geometry. We solve the problem in dimension $n = 2$ for axisymmetric convex bodies, by proving that the solution is the square. To that aim we first reduce our problem to a reverse Faber-Krahn inequality for axisymmetric convex octagons, and then we identify an optimal octagon with the one which degenerates into a square. For this latter challenge, we employ a hybrid method inspired by the Polymath blog by Tao, which is based on the joint use of theoretical arguments to settle octagons lying in computable "neighborhoods" of the square, and of a numerical argument (rigorously working thanks to the monotonicity by inclusions of the involved functionals) to settle octagons lying outside the confidence zones.

1. Introduction

Let $\mathcal{K}^n$ be the class of convex bodies in $\mathbb{R}^n$ (convex compact sets with nonempty interior), and let $\mathcal{K}^n_*$ be the subclass of centrally symmetric convex bodies. Given any $K \in \mathcal{K}^n$ with the origin in its interior, the *polar body* of K is defined as $K^o := \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \ \forall x \in K\}$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$. For $K \in \mathcal{K}^n_*$, the quantity

$$|K||K^o|$$

is called the *volume product* of K. This quantity turns out to be invariant under invertible linear transformations, as it follows immediately from the equalities $(T(K))^o = (T^t)^{-1}(K^o)$ and $|T(K)| = |\det T||K|$, holding for every $T \in GL_n$.
In 1939 Mahler conjectured that, when K varies in $\mathcal{K}^n_*$, the maximum of the volume product should be attained at the ball, whereas its minimum should be attained at the n-cube. Clearly, all the linear images of these domains should be optimal as well.
For non-centered bodies, the notion of volume product has to be generalized into

$$\min_{x \in \mathrm{int}(K)} |K||(K - x)^o|$$

(so that it becomes invariant under affinities), and the analogous conjecture tells that the extremal domains are balls and n-simplexes (and their affine images).

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The upper bound inequality was proved in [7, 43], and is known as *Blaschke-Santaló inequality*: denoting by B a generic ball, it holds

$$\min_{x\in\mathrm{int}(K)} |K||(K-x)^o| \le |B||B^o| \qquad \forall K\in\mathcal{K}^n\,;$$

the developments of this important inequality in convex affine geometry are still object of investigation, see for instance [37, 33, 20, 11, 6, 41] .
The lower bound inequality, which is called *the Mahler conjecture*, is still open and is one of the most fascinating and actual problems in Convex Geometry. Up to now, it has been proved only for $n=2$ and for K belonging to some restricted classes of convex bodies in higher dimensions. For $n=2$ it was proved in [34] by Mahler himself, who also proved that, when K varies in $\mathcal{K}^n$, the volume product stays bounded by strictly positive constants, depending only on the space dimension n. Some special classes of convex bodies where the conjecture has been settled are: unconditional bodies [42], zonoids [40], and polyhedra with a small number of vertices [32]. For general convex bodies, estimates from below for the volume product have been proved by Bourgain-Milman [12] and Kuperberg [30]. Further quite active related research directions include the investigation of local minima [38, 29, 25], stability results [10], and functional forms of the inequality [21]. For more references and comments on Mahler's conjecture, we refer to [44, Sec. 10.7] and [45, Sec. 3.8].
Let us now turn to what we like to call the "variational side" of these geometric inequalities. Aim of this paper is to launch and attack the study of inequalities involving the notion of polarity for functionals other than volume, which are not purely geometric, but are associated with some elliptic variational problem, such as the first Dirichlet eigenvalue of the Laplacian, the torsional rigidity, or the Newtonian capacity. In fact, on the class $\mathcal{K}^n$ these functionals behave in a similar way as volume under different aspects, including: monotonicity with respect to inclusions, homogeneity with respect to homotecies, response under Schwarz symmetrization [26], validity of a Brunn-Minkowski type inequality [8, 9, 17], existence and uniqueness for the Minkowski problem [28, 27, 18]. For this reason, it seems somehow natural to investigate the validity of upper or lower bounds for "variational Mahler products", namely functionals of the type $f(K)f(K^o)$, where f is one of the above mentioned energy costs. To the best of our knowledge, any result in this direction should sound completely new in the broad framework of isoperimetric or spectral inequalities (see for instance [2, 3, 46, 47, 15, 16, 19, 22, 23, 24, 31, 35, 39]).
In this paper, we focus our attention on the case of the principal frequency, and we study extremum problems for the following functional, that we call *the λ_1-product*:

$$\lambda_1(K)\lambda_1(K^o)\,;$$

here and in the sequel, $\lambda_1(K)$ stands for the first eigenvalue of the Laplacian on the relative interior of K with Dirichlet condition $u=0$ on ∂K.
As a first result, we establish an analogue of the Blaschke-Santaló inequality, which in the centrally symmetric version states that balls solve the minimization problem

$$\inf\Big\{\lambda_1(K)\lambda_1(K^o)\ :\ K\in\mathcal{K}^n_*\Big\} \tag{1}$$

(see Theorem 1). The fact that balls are minimizers for the λ_1-product whereas they are maximizers for the volume product, is naturally due to the monotonicity under inclusions of λ_1, which is opposite with respect to the case of volume. The proof of Blaschke-Santaló inequality for λ_1 is quite simple and relies on the combination of Blaschke-Santaló

inequality for the volume product with the classical Faber-Krahn inequality, that we recall under the form (see *e.g.* [26])

$$\lambda_1(K)|K|^{\frac{2}{n}} \geq \lambda_1(B)|B|^{\frac{2}{n}} \qquad \forall K \in \mathcal{K}^n\,. \tag{2}$$

On the other hand, the maximization problem for the λ_1-product appears as a counterpart of Mahler conjecture, and it seems to be equally or even more delicate than the latter is. One of the reasons is that, to the best of our knowledge, there is no specific relation between the first Dirichlet eigenfunction of K^o and the one of K, nor more generally a canonical way to produce a trial function in $H^1_0(K^o)$ starting from an element of $H^1_0(K)$. Another basic fact is that the λ_1-product is no longer invariant under invertible linear transformations, and the problem of maximizing it over $\mathcal{K}^n$ or over $\mathcal{K}^n_*$, taken as such, is ill posed, since the λ_1-product tends to $+\infty$ along a sequence of thinning domains (*e.g.*, in dimension $n = 2$, along a sequence of thinning rectangles). Thus, if we work to fix the ideas in the centrally symmetric setting, in order to keep the supremum finite and recover the existence of a maximizing domain, it is necessary to preliminarily minimize over all the invertible linear images of K. In other words, the correct formulation of the problem reads

$$\sup\Big\{\inf_{T\in GL_n} \lambda_1(T(K))\lambda_1(T(K)^o) \;:\; K \in \mathcal{K}^n_*\Big\}\,. \tag{3}$$

We believe that, as it happens for the volume product, also the λ_1-product is, loosely speaking, a "measure of roundedness" for convex bodies; in particular, we conjecture that the solution to the above problem is the n-cube. In this paper we prove such conjecture in space dimension $n = 2$ and under the requirement the admissible convex bodies satisfy the additional condition of being *axisymmetric*.
Throughout the paper, we denote by $\mathcal{K}^2_\sharp$ the class of axisymmetric bodies in $\mathcal{K}^2$.
Our main result (see Theorem 9) states that the square solves

$$\sup\Big\{\inf_{T\in D_2} \big[\lambda_1(T(K))\lambda_1((T(K))^o)\big] \;:\; K \in \mathcal{K}^2_\sharp\Big\}\,, \tag{4}$$

where D_2 denotes the class of invertible diagonal transformations of $\mathbb{R}^2$ into itself.
In a similar way as problem (1) is related to the Faber-Krahn inequality (2), problem (3) (or its simplest version (4)) leads in a natural way to wonder about the validity of a "reverse form" of the Faber-Krahn inequality. For analogous reasons as above, such a reverse form should be searched modulo invertible linear transformations, namely by studying the problem

$$\sup\Big\{\inf_{T\in GL_n} \big[\lambda_1(T(K))|T(K)|^{\frac{2}{n}}\big] \;:\; K \in \mathcal{K}^n_*\Big\}\,,$$

which we expect to be solved by the n-cube. So far, the only antecedent about reverse isoperimetric-type inequalities exists in a purely geometric context: it is the beautiful result due to Ball (see [4]) which asserts that, for every $K \in \mathcal{K}^n_*$, there is a linear image $T(K)$ such that the isoperimetric quotient $|\partial T(K)|/|T(K)|^{\frac{n-1}{n}}$ is not larger than the corresponding expression for a n-cube. In other words, the n-cube solves

$$\sup\Big\{\inf_{T\in GL_n} \big[|\partial(T(K))||T(K)|^{-\frac{n-1}{n}}\big] \;:\; K \in \mathcal{K}^n_*\Big\}\,.$$

(A similar statement holds on $\mathcal{K}^n$ with the n-simplex as optimal domain).
Our route in order to prove Theorem 9 is as follows. Inspired by a proof of Mahler conjecture for unconditional convex bodies due to Meyer [36], we show that a sufficient condition

in order that the square solves problem (4) is that it solves the following "restricted version" of the reverse Faber-Krahn inequality

$$\sup\Big\{\big[\lambda_1(\Omega)|\Omega|\big]\ :\ \Omega\in\mathcal{O}\Big\}, \tag{5}$$

where $\mathcal{O}$ denotes the class of convex axisymmetric *octagons* having four of their vertices lying on the axes at the same distance, say 1, from the origin (see the key Proposition 10). We are then reduced to prove that the square is an optimal domain in problem (5). The proof of such result (which is stated as Theorem 12) turns out to be quite delicate, and to it is devoted most part of the paper. To solve the problem we have tracked the promising approach proposed by Tao on the Polymath Blog, more precisely within the discussion about the hot-spot conjecture for acute-angled triangles (see *http://polymathprojects.org/2013/08/09/polymath7-research-thread-5-the-hot-spots-conjecture/)*. Such approach may be called *hybrid method*, as it involves both theoretical and numerical tools, and takes advantage of different arguments in order to manage with a multi-aspect problem. To explain better what we mean, let us look at (5) as a two-parameter problem. Notice indeed that, denoting by Q_- the square with vertices $(\pm1,0)$ and $(0,\pm1)$, and by Q_+ the square with vertices $(\pm1,\pm1)$, for any convex octagon in the class $\mathcal{O}$, four of its vertices are fixed at the vertices of Q_-, and by symmetry the remaining four are determined by one of them, which is free to move in the region $(Q_+\setminus Q_-)\cap(\mathbb{R}_+\times\mathbb{R}_+)$. The coordinates of this vertex $x=(x_1,x_2)$ are the two parameters describing the class $\mathcal{O}$ in our problem.
Then the theoretical part of our proof consists in showing that the product $\lambda_1(\Omega)|\Omega|$ is smaller than its value for a square (which equals $2\pi^2$) provided the octagon Ω is "sufficiently close" to Q_+ or Q_-. Two main features have to be highlighted in this respect. First, we can precise exactly "how much close" to $Q_\pm$ an octagon Ω must be in order that our theoretical proof works; in other terms, the "confidence zones" where we are able to prove directly the inequality are perfectly *computable*. Second, the proof has to be different according to the way the octagon approaches the square; essentially three regimes can be identified, each of which demands a completely different strategy:

- Regime I, when x is close to the vertex $(1,1)$ of Q_+: in this case our strategy relies on a continuous Steiner de-symmetrization argument, combined with the construction, for hexagons close to Q_+, of a test function enjoying some special features, obtained via an affine deformation of the first eigenfunction of Q_+.
- Regime II, when x is close to a boundary point of Q_- which is not a vertex: in this case we construct a good test function by using an affine deformation of the first eigenfunction of Q_-, and we proceed via explicit computations.
- Regime III, when x is close to the vertex $(0,1)$ (or to its symmetric $(0,1)$): this is the most delicate case, where we need to apply a cut-off argument inspired by Alt-Caffarelli, and take into account the slope of the line joining x with $(0,1)$, which enters into the game in a subtle way.

Once determined the confidence zones, the final part of the work consists in proving numerically that the inequality $\lambda_1(\Omega)|\Omega|\le 2\pi^2$ holds true also outside them. At this stage, the crucial idea which makes possible to obtain a mathematically rigorous proof via a numerical approach is based on two arguments. First, observe that it is enough to check that $\lambda_1(\Omega')|\Omega''|\le 2\pi^2$, where Ω' and Ω'' are octagons parametrized by the nodes of a sufficiently fine mesh such that $\Omega'\subset\Omega\subset\Omega''$. The reason why this check is sufficient is simply the monotonicity of λ_1 and volume with respect to inclusions. Second, the inequality we shall prove is in fact $\lambda_1^{num}(\Omega')|\Omega''|\le 2\pi^2$, where $\lambda_1^{num}(\Omega')$ is a numerically

computed approximation of $\lambda_1(\Omega')$ satisfying $\lambda_1(\Omega') < \lambda_1^{num}(\Omega')$. This last inequality, which plays a key role, is insured by the choice of the numerical method, as detailed in the last section of the paper.
The extension of our results to the non axisymmetric case or to higher space dimensions seems to be an interesting and challenging objective for further research.

Outline of the paper.
In Section 2 we prove Blaschke-Santaló inequality for λ_1 (see Theorem 1), and we state our main result on Mahler problem for λ_1 in the axisymmetric planar case (see Theorem 9), after proving its well-posedness in a more general setting (see Proposition 5).
In Section 3, we show how Theorem 9 follows provided the square is optimal in the reverse Faber-Krahn for octagons, namely provided it solves problem (5) (see Proposition 10). We also give some numerical experiments in favor of the optimality of the square for problem (5), and then we state it in Theorem 12, whose proof is subdivided in the subsequent sections.
In Sections 4 and 5, we determine computable regions of validity of the reverse Faber-Krahn inequality for octagons, respectively close to the exterior square Q_+ (regime I) and close to the interior square Q_- (regimes II and III).
In Section 6 we describe the numerical proof outside the confidence zones.

2. Blaschke-Santaló inequality and Mahler problem for λ_1

We begin by considering the maximization problem for the λ_1-product, in the centrally symmetric setting.

Theorem 1. (Blaschke-Santaló inequality for λ_1, centrally symmetric case)
Let B be a ball. Then, for every $K \in \mathcal{K}_*^n$, it holds

$$\lambda_1(K)\lambda_1(K^o) \geq \lambda_1(B)\lambda_1(B^o)\,. \tag{6}$$

Proof. By the Faber-Krahn inequality (see for instance [26, Section 3.2]), for all $K \in \mathcal{K}^n$ we have

$$\lambda_1(K) \geq \lambda_1(K^*) \qquad \text{and} \qquad \lambda_1(K^o) \geq \lambda_1\big((K^o)^*\big)\,,$$

where K^* and $(K^o)^*$ denote respectively the ball with same volume as K and K^o. Then in order to prove (6) it is enough to show that

$$\lambda_1\big((K^o)^*\big) \geq \lambda_1\big((K^*)^o\big) \qquad \forall K \in \mathcal{K}_*^n\,.$$

By monotonicity of $\lambda_1(\cdot)$, the above inequality is satisfied provided

$$(K^o)^* \subseteq (K^*)^o \qquad \forall K \in \mathcal{K}_*^n\,. \tag{7}$$

Denote by B_r the ball centered at the origin with radius $r > 0$, and by ω_n the Lebesgue measure of the unit ball in $\mathbb{R}^n$. By definition, we have $(K^o)^* = B_{r_1}$ and $(K^*)^o = B_{r_2}$, where the radii r_1 and r_2 are defined by the equalities

$$|K^o| := \omega_n r_1^n \qquad \text{and} \qquad |K| = \frac{\omega_n}{r_2^n}$$

(note that the second equality holds since $K^* = B_{r_2}^o = B_{\frac{1}{r_2}}$). By the Blaschke-Santaló inequality for centrally symmetric bodies [7, 43], we know that $|K|\,|K^o| \leq \omega_n^2$, so we infer that

$$\omega_n^2\,\frac{r_1^n}{r_2^n} \leq \omega_n^2$$

namely $r_1 \le r_2$, which implies (7) and achieves the proof. □

Remark 2. By inspection of the above proof, it is readily seen that inequality (6) can be strengthened into

$$\inf_{T\in GL_n} \lambda_1(T(K))\lambda_1\big((T(K))^o\big) \ge \lambda_1(B)\lambda_1(B^o) \qquad \forall K \in \mathcal{K}^n_* \,.$$

Indeed, it is enough to replace the body K in the proof of Theorem 1 by $K' := T(K)$, T being an arbitrary element of GL_n, and use the inequality $|K'||(K')^o| \le \omega_n^2$. On the other hand, if one wants to allow also non-centered convex bodies, the inequality must be stated as

$$\sup_{x\in \mathrm{int}(K)} \inf_{T\in GL_n} \lambda_1(T(K-x))\lambda_1\big((T(K-x))^o\big) \ge \lambda_1(B)\lambda_1(B^o) \qquad \forall K \in \mathcal{K}^n \,. \tag{8}$$

In this case one has to replace the body K in the proof of Theorem 1 by $K'' := T(K-s(K))$, T being an arbitrary element of GL_n and $s(K)$ the so-called *Santaló point* of K (which is defined as the unique solution to the minimization problem $\min\{|(K-x)^o| : x \in \mathrm{int}(K)\}$), and then invoke the general version of Blaschke-Santaló inequality $|K''||(K'')^o| \le \omega_n^2$.

We now turn to the more challenging extremum problem for the λ_1-product, the maximization one; we start starting with a simple but important observation, already mentioned in the Introduction:

Remark 3. There holds

$$\sup_{K\in\mathcal{K}^n} \lambda_1(K)\lambda_1(K^o) = \sup_{K\in\mathcal{K}^n_*} \lambda_1(K)\lambda_1(K^o) = +\infty \,.$$

To see this, one has to consider the asymptotic behavior of the λ_1-product along a sequence of (possibly centrally symmetric) thinning domains. For instance, in dimension $n = 2$, it is enough to take a sequence of thinning rectangles, $K_h := [-1, 1] \times [-\frac{1}{h}, \frac{1}{h}]$, and check via some straightforward computations that $\lim_h \lambda_1(K_h)\lambda_1((K_h)^o) = +\infty$.

However, the well-posedness of the maximization problem for the λ_1-product can be easily recovered by passing preliminarily to the infimum over a suitable family of images of K. To be more precise, we rely on the following abstract lemma, where we denote by $\mathcal{K}^n_\sharp$ the class of unconditional bodies, namely bodies which are symmetric with respect to all the coordinate hyperplanes of a fixed frame.

Lemma 4. *Let $J(\cdot)$ be a shape functional defined on $\mathcal{K}^n$. Assume that J is upper semi-continuous with respect to the Hausdorff distance, and that it is invariant under one of the following family of transformations:*

(i) *invertible affine transformations;*
(ii) *invertible linear transformations;*
(iii) *invertible diagonal transformations.*

The J attains a supremum, respectively: (i) over $\mathcal{K}^n$; (ii) over $\mathcal{K}^n_*$; (iii) over $\mathcal{K}^n_\sharp$.

Proof. If J is invariant under transformations respectively as in (i), (ii), and (iii), let $\{K_h\}$ be a maximizing sequence for J over $\mathcal{K}^n$, over $\mathcal{K}^n_*$, and over $\mathcal{K}^n_\sharp$. By John's Lemma (see

e.g. [5, Lecture 3]), for every h there exists a ball B_h and an affine transformation $\widetilde{K}_h$ of K_h such that $B_h \subseteq \widetilde{K}_h \subseteq nB_h$ and hence

$$\frac{B_h}{|\widetilde{K}_h|} \subseteq \frac{\widetilde{K}_h}{|\widetilde{K}_h|} \subseteq n\frac{B_h}{|\widetilde{K}_h|}\,. \tag{9}$$

We point out that $\widetilde{K}_h$ is constructed as $T_h(K_h)$, T_h being an affine transformation which maps the ellipsoid of maximal value contained into K_h into a ball. In particular, if K_h belongs respectively to $\mathcal{K}^n$, $\mathcal{K}^n_*$, and $\mathcal{K}^n_\sharp$, we have that $\widetilde{K}_h$ is an affine, linear, or diagonal image of K_h. Then, by the invariance assumption made on J, we can assert that $\frac{\widetilde{K}_h}{|\widetilde{K}_h|}$ is still a maximizing sequence respectively over $\mathcal{K}^n$, $\mathcal{K}^n_*$, and $\mathcal{K}^n_\sharp$.
Denoting by r_h the radius of the ball $\frac{B_h}{|\widetilde{K}_h|}$, and by ω_n the measure of the unit ball in $\mathbb{R}^n$, by (9) we have

$$\omega_n r_h^n \le 1 \le n\omega_n r_h^n\,.$$

Thus the sequence $\{r_h\}$ is bounded from above and from below, and the inclusions (9) show that all the bodies $\frac{\widetilde{K}_h}{|\widetilde{K}_h|}$ are contained into a ball of fixed radius and contain a ball of fixed radius. By Blaschke selection Theorem (see *e.g.* [44, Theorem 1.8.6]), we infer that after passing to a (not relabeled) subsequence, $\frac{\widetilde{K}_h}{|\widetilde{K}_h|}$ converge in Hausdorff distance to a non degenerated limit body K, which belongs respectively to $\mathcal{K}^n$, $\mathcal{K}^n_*$, and $\mathcal{K}^n_\sharp$ (since any of these classes is closed under Hausdorff convergence). By the upper semicontinuity assumption made on J, it holds

$$J(K) \ge \limsup J\Big(\frac{\widetilde{K}_h}{|\widetilde{K}_h|}\Big) = \sup_{\mathcal{K}^n} J,$$

hence K is a maximizer for J respectively over $\mathcal{K}^n$, $\mathcal{K}^n_*$, and $\mathcal{K}^n_\sharp$. □

By applying the above lemma we readily get

Proposition 5. (Mahler problem for λ_1, centrally symmetric and unconditional cases) *The following problems admit a solution:*

$$\sup\Big\{\inf_{T\in GL_n} \lambda_1(T(K))\lambda_1((T(K))^o) \ : \ K\in\mathcal{K}^n_*\Big\}, \tag{10}$$

$$\sup\Big\{\inf_{T\in D_n} \lambda_1(T(K))\lambda_1((T(K))^o) \ : \ K\in\mathcal{K}^n_\sharp\Big\}. \tag{11}$$

Proof. If J is defined as the infimum of $\lambda_1(T(K))\lambda_1((T(K))^o)$, respectively over $T\in GL_n$ and over $T\in D_n$, then J satisfies the assumptions of Lemma 4 (ii) and (iii), and the result follows. □

Remark 6. If one would like to consider the λ_1-Mahler problem for non-centered convex bodies, few words of warning are in order. Indeed, by analogy with (10)-(11), one could erroneously pass to the infimum of $\lambda_1(T(K))\lambda_1((T(K))^o)$ for T ranging over all affine transformations or $\mathbb{R}^n$: the result would be simply zero, due to translations and more precisely to the fact that $\lambda_1((K-x)^o)$ is infinitesimal as $x\to\partial K$. In spite, one should pass to the *supremum* over translations and to the *infimum* over invertible linear transformations (in a similar way as done in (8)). However, it is not immediate to get the

well-posedness of the corresponding problem via Lemma 4, because the upper semicontinuity is no longer straightforward. We do not enter more into details in this respect, because in the remaining of the paper we deal just with symmetric bodies.

Remark 7. A very rough upper bound for the supremum in (11) can be found via John Lemma: for every $K \in \mathcal{K}^n_*$, there exists $T \in GL_n$ and a ball B_r of radius r such that

$$B_r \subseteq T(K) \subseteq B_{\sqrt{n}r} \qquad B_{\frac{1}{\sqrt{n}r}} \subseteq (T(K))^o \subseteq B_{\frac{1}{r}}\,,$$

which by monotonicity of $\lambda_1(\cdot)$ with respect to inclusions implies

$$\lambda_1(T(K))\lambda_1((T(K))^o) \leq \lambda_1(B_r)\lambda_1(B_{\frac{1}{\sqrt{n}r}})\,.$$

For instance, in dimension $n = 2$, this tells that the supremum in (11) is bounded above by $2j^4$, where $j \approx 2.405$ is the first zero of the Bessel function J_0 (*cf.* [26, Section 1.2.5]).

Remark 8. For all the results of this section, similar statements continue to hold, with unaltered proofs, if λ_1 is replaced by the torsional rigidity or the Newtonian capacity. It is enough to take care to reverse all the inequalities, due to the different monotonicity of these functionals with respect to inclusions.

In the remaining of the paper we deal with Mahler problem for λ_1 in case of planar axisymmetric convex bodies, and we prove:

Theorem 9. *In dimension $n = 2$, problem* (11) *is solved by the square.*

3. From Mahler problem for λ_1 to a reverse Faber-Krahn inequality for convex octagons

In this section we provide a sufficient condition for the validity of Theorem 9, under the form of a reverse Faber-Krahn inequality for a family of convex octagons.
Denote by $\mathcal{O} \subset \mathcal{K}^2_\sharp$ the class of convex axisymmetric octagons having four of their vertices lying on the axes at the same distance from the origin. If $\Omega \in \mathcal{O}$ has four vertices at $(\pm\ell, 0)$ and $(0, \pm\ell)$, the remaining four vertices of Ω will be of the form $(\pm x_1, \pm x_2)$, with (x_1, x_2) belonging to the triangular region $\big\{(x_1, x_2) \in [0, \ell]^2 \,:\, x_2 \geq \ell - x_1\big\}$. In particular, when (x_1, x_2) agrees with the point (ℓ, ℓ), or falls upon the line segment $\big\{x_1 \in [0, \ell]\,,\ x_2 = \ell - x_1\big\}$, the corresponding octagon degenerates into a square. Note that, for any square Q, it holds

$$\lambda_1(Q)|Q| = 2\pi^2\,.$$

We have:

Proposition 10. *Assume that the square solves the maximization problem*

$$\sup_{\Omega \in \mathcal{O}} \lambda_1(\Omega)|\Omega|\,, \tag{12}$$

namely there holds

$$\lambda_1(\Omega)|\Omega| \leq \lambda_1(Q)|Q| = 2\pi^2 \qquad \forall \Omega \in \mathcal{O}\,. \tag{13}$$

Then the square solves the maximization problem

$$\sup\Big\{\inf_{T \in D_2}\big[\lambda_1(T(K))\lambda_1(T(K)^o)\big] \;:\; K \in \mathcal{K}^2_\sharp\Big\}\,, \tag{14}$$

namely for every $K \in \mathcal{K}^2_\sharp$ there exists $T \in D_2$ such that

$$\lambda_1(T(K))\lambda_1(T(K)^o) \leq \inf_{T \in D_2}\big[\lambda_1(T(Q))\lambda_1(T(Q)^o)\big] = \frac{\pi^4}{2}\,. \tag{15}$$

In order to prove Proposition 10, we begin by observing that the last equality in (15) is an immediate consequence of the optimality of the square in the Faber-Krahn inequality for quadrilaterals. Indeed it holds:

Lemma 11. *Let $Q \subset \mathbb{R}^2$ be a square, and let $T \in D_2$. Then*

$$\lambda_1(T(Q))\lambda_1(T(Q)^o) \geq \lambda_1(Q)\lambda_1(Q^o) = \frac{\pi^4}{2}\,.$$

Proof. For any $T \in D_2$, by using the Faber-Krahn inequality for quadrilaterals (see *e.g.* [26, Section 3.3]), and the invariance of the volume product under invertible linear transformations, we get

$$\begin{aligned}\lambda_1(T(Q))\lambda_1(T(Q)^o) &= \frac{\lambda_1(T(Q))|T(Q)|\lambda_1((T(Q))^o)|(T(Q))^o|}{|T(Q)||(T(Q))^o|}\\ &\geq \frac{\lambda_1(Q)|Q|\lambda_1(Q^o)|Q^o|}{|Q||Q^o|} = \lambda_1(Q)\lambda_1(Q^o)\end{aligned}$$

□

Proof of Proposition 10. In view of Lemma 11, we need to show that, for every $K \in \mathcal{K}^2_\sharp$, there exists $T \in D_2$ such that

$$\lambda_1(T(K))\lambda_1(T(K)^o) \leq \frac{\pi^4}{2}\,. \tag{16}$$

Let $e_1 = (1,0)$ and $e_2 = (0,1)$, and let $\langle e_1\rangle, \langle e_2\rangle$ be the coordinate axes.
If $K \cap \langle e_i\rangle = [-\ell_i, \ell_i]$ for $i = 1,2$, we are going to show that (16) holds true by choosing as an element $T \in D_2$ the map

$$T(x_1,x_2) := \big(\frac{\ell_2}{\ell_1}x_1, x_2\big) \qquad \forall (x_1,x_2) \in \mathbb{R}^2\,. \tag{17}$$

Note that the transformed body $T(K)$ satisfies the equality condition

$$|T(K) \cap \langle e_1\rangle| = |T(K) \cap \langle e_2\rangle| (= \ell_2)\,.$$

Moreover, since $K^o \cap \langle e_i\rangle = -[\frac{1}{\ell_i}, \frac{1}{\ell_i}]$ for $i = 1,2$, the dual transformed body $(T(K))^o = (T^t)^{-1}(K^o)$ (where $(T^t)^{-1}$ is the transformation $(T^t)^{-1}(y_1,y_2) = \big(\frac{\ell_1}{\ell_2}y_1, y_2\big)$) satisfies the same kind of equality condition, namely

$$|(T(K))^o \cap \langle e_1\rangle| = |(T(K))^o \cap \langle e_2\rangle| (= \frac{1}{\ell_2})\,.$$

Assume for a moment we are able to prove the following

Claim: if $Z \in \mathcal{K}^2_\sharp$ satisfies $|Z \cap \langle e_1\rangle| = |Z \cap \langle e_2\rangle| =: 2\ell$, it holds

$$\lambda_1(Z)(x_1+x_2) \leq \frac{\pi^2}{\ell} \qquad \forall x = (x_1,x_2) \in Z\,. \tag{18}$$

Applying such claim also to the polar body Z^o (which satisfies $|Z^o \cap \langle e_1\rangle| = |Z^o \cap \langle e_2\rangle| = \frac{2}{\ell}$), we get

$$\lambda_1(Z^o)(y_1+y_2) \leq \ell\,\pi^2 \qquad \forall y = (y_1,y_2) \in Z^o\,. \tag{19}$$

The inequalities (18) and (19) imply respectively that

$$v := \Big(\frac{\lambda_1(Z)}{\frac{\pi^2}{\ell}}, \frac{\lambda_1(Z)}{\frac{\pi^2}{\ell}}\Big) \in Z^o \qquad \text{and} \qquad w := \Big(\frac{\lambda_1(Z^o)}{\ell\pi^2}, \frac{\lambda_1(Z^o)}{\ell\pi^2}\Big) \in Z\,.$$

Taking the scalar product of v and w we infer

$$\frac{\lambda_1(Z)\lambda_1(Z^o)}{\pi^4}+\frac{\lambda_1(Z)\lambda_1(Z^o)}{\pi^4}\le 1\,.$$

We have thus obtained that the inequality $\lambda_1(Z)\lambda_1(Z^o)\le\frac{\pi^4}{2}$ is satisfied for every $Z\in\mathcal{K}^2_*$ which satisfies the equality condition $|Z\cap\langle e_1\rangle|=|Z\cap\langle e_2\rangle|$. By applying such inequality to $Z=T(K)$, K being an arbitrary element of $\mathcal{K}^2_\sharp$ and T the transformation chosen as in (17), we obtained that (16) is satisfied.
In order to achieve the proof, it only remains to prove the above claim. To that aim, we are going to exploit the assumption (13). Let $Z\in\mathcal{K}^2_\sharp$ with $|Z\cap\langle e_1\rangle|=|K\cap\langle e_2\rangle|=2\ell$, and let $x=(x_1,x_2)\in Z$ be fixed. In order to prove (18), we may assume that x_1 and x_2 are nonnegative. Clearly, by convexity, Z turns out to contain the octagon with vertices at $(\pm x_1,\pm x_2)$, $(\pm\ell,0)$, $(0,\pm\ell)$. Denote such octagon by $\Omega^\ell_{(x_1,x_2)}$. By monotonicity of λ_1 with respect to inclusion, it holds, $\lambda_1(Z)\le\lambda_1(\Omega^\ell_{(x_1,x_2)})$. Moreover, an immediate computation gives

$$|\Omega^\ell_{(x_1,x_2)}|=4\Big(\frac{1}{2}\ell x_1+\frac{1}{2}\ell x_2\Big)=2\ell(x_1+x_2)\,.$$

Therefore (18) holds true, since

$$\lambda_1(Z)2\ell(x_1+x_2)\le\lambda_1(\Omega^\ell_{(x_1,x_2)})2\ell(x_1+x_2)=\lambda_1(\Omega^\ell_{(x_1,x_2)})\,|\Omega^\ell_{(x_1,x_2)}|\le 2\pi^2\,,$$

where the last inequality follows from assumption (13). □

Thanks to Proposition 10, in order to prove Theorem 9 we are reduced to study the simpler problem (12). The different advantages of studying problem (12) in place of problem (14) are self-evident: we have get rid of polarity, we only have to deal with convex octagons, and we can also easily perform some numerical tests. Let us report below some numerical experiments on the validity of inequality (13). Since the product $\lambda_1(\Omega)|\Omega|$ is invariant by scaling, with no loss of generality one can restrict the analysis to octagons in $\mathcal{O}$ having four vertices at the points $(\pm1,0)$ and $(0,\pm1)$; the remaining four vertices will be of the form $(\pm x_1,\pm x_1)$, with (x_1,x_2) lying in the triangular region

$$\Delta:=\big\{(x_1,x_2)\in\big[0,1\big]^2\ :\ x_2\ge 1-x_1\big\}\,. \tag{20}$$

For $(x_1,x_2)\in\Delta$, set

$$\Omega_{(x_1,x_2)}:=\text{ the octagon with vertices }(\pm1,0)\,,\ (0,\pm1)\,,\ (\pm x_1,\pm x_2)\,. \tag{21}$$

(note that $\Omega_{(x_1,x_2)}$ reduces to a hexagon or to a square for $(x_1,x_2)\in\partial\Delta$), and consider the map

$$\mathcal{E}(x_1,x_2):=\lambda_1(\Omega_{(x_1,x_2)})|\Omega_{(x_1,x_2)}|\,. \tag{22}$$

With this notation, inequality (13) states that the maximum of the above map $\mathcal{E}$ over the region Δ is equal to $2\pi^2$ and it is attained when the point (x_1,x_2) lies either at the vertex P or along the segment S defined by

$$P:=(1,1)\quad\text{and}\quad S:=\big\{(x_1,x_2)\in\big[0,1\big]^2\ :\ x_2\ge 1-x_1\big\}\,; \tag{23}$$

correspondingly, Ω reduces respectively to

$$Q_+:=\text{ the square with vertices }(\pm1,\pm1) \tag{24}$$

$$Q_-:=\text{ the square with vertices }(\pm1,0)\text{ and }(0,\pm1)\,. \tag{25}$$

We enclose hereafter two plots of the map $\mathcal{E}$ on such region, which bring some evidence to the truthfulness of the inequality (13).
In fact we are going to prove the following result, which in view of Proposition 10 implies also Theorem 9:

Theorem 12. *Inequality* (13) *holds true (and in addition it is strict unless* Ω *is a square), and hence the unique solution to problem* (12) *is the square.*

As announced in the Introduction, in order to obtain Theorem 12 we pursue a hybrid theoretical-numerical method, which is developed in detail in the next sections.

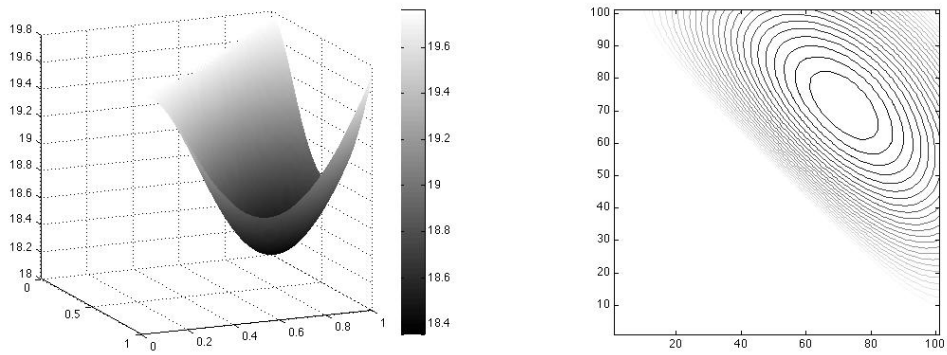

FIGURE 1. Plot of the map $\mathcal{E}$ and its level sets over Δ

4. Confidence zone near the point P

In this section we prove that the function $\mathcal{E} = \mathcal{E}(x_1, x_2)$ introduced in (22) assumes values smaller than $2\pi^2$ in a computable neighborhood of the point $P = (1, 1)$.
In order to prepare our estimate from above the map $\mathcal{E}$ near P, we need the following definition and the subsequent lemma.
For every $\Omega \in \mathcal{K}^2_\sharp$, let $\widetilde{H}(\Omega)$ be the subspace of functions $u \in H^1_0(\Omega)$ which satisfy the following conditions

$$u(x_1, x_2) = u(-x_1, x_2) = u(x_1, -x_2) = u(-x_1, -x_2) \quad \forall\, (x_1, x_2) \in \Omega\,; \tag{26}$$

$$\tilde{u}(x_1, 0) = \tilde{u}(0, x_1) \quad \forall\, (x_1, 0) \in \Omega \cap (\mathbb{R} \times \{0\})\,; \tag{27}$$

$$\tilde{u}(x_1, x_2) \geq \tilde{u}(x_1 + x_2, 0) \quad \forall\, (x_1, x_2) \in \Omega \cap (\mathbb{R}_+)^2\,; \tag{28}$$

in (27) and (28), $\tilde{u}$ stands for the trace of u, respectively on the coordinate axes and on the line segment $y = -x + x_1 + x_2$.

Definition 13. We define the modified eigenvalue $\widetilde{\lambda}_1(\Omega)$ by

$$\widetilde{\lambda}_1(\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega |u|^2\, dx} \ : \ u \in \widetilde{H}(\Omega) \right\}$$

Remark 14. It is immediate from the above definition that the inequality $\lambda_1(\Omega) \leq \widetilde{\lambda}_1(\Omega)$ holds for every $\Omega \in \mathcal{K}^2_\sharp$.

Lemma 15. *For any $\varepsilon \in (0,1)$, let $\Omega_{(1-\varepsilon,1)}$ be the hexagon defined according to* (21)*. Then it holds*

$$\widetilde{\lambda}_1\big(\Omega_{(1-\varepsilon,1)}\big) \le \frac{\frac{\pi^2}{16}+\frac{\varepsilon}{4}+\frac{\pi^2}{16}\Big(\frac{1-\varepsilon+\varepsilon^2}{1-\varepsilon}\Big)}{\frac{1}{4}+\varepsilon\Big(\frac{1}{\pi^2}-\frac{1}{8}\Big)}\,. \tag{29}$$

Proof. In order to obtain inequality (29), we are going to exploit a suitable test function $U \in \widetilde{H}\big(\Omega_{(1-\varepsilon,1)}\big)$, defined as a deformation of the first Dirichlet eigenfunction of the Laplacian on Q_+. Recall that Q_+ denotes the square with vertices $(\pm1,\pm1)$ and that, up to constant multiples, its first eigenfunction is given by

$$u(x_1,x_2) := \cos\Big(\frac{\pi x_1}{2}\Big)\cos\Big(\frac{\pi x_2}{2}\Big)\,. \tag{30}$$

Let $T : Q_+ \cap \big\{x_1 \ge 0, x_2 \ge 0\big\} \to \Omega_{(1-\varepsilon,1)} \cap \big\{X_1 \ge 0, X_2 \ge 0\big\}$ be the transformation defined as follows:

$$(X_1,X_2) = T(x_1,x_2) := \begin{cases} (x_1,x_2) & \text{if } x_1+x_2 \le 1 \\ \big((1-\varepsilon)x_1 - \varepsilon x_2 + \varepsilon, x_2\big) & \text{if } x_1+x_2 \ge 1 \end{cases}$$

Note that, for $x_1+x_2 \ge 1$, T is uniquely determined as the affine map which keeps the points $(1,0)$ and $(0,1)$ fixed, and moves the point $(1,1)$ into $(1-\varepsilon,1)$. Then we take as a test function U the axisymmetric function defined on $\Omega_{(1-\varepsilon,1)} \cap \big\{X_1 \ge 0, X_2 \ge 0\big\}$ by

$$U(X_1,X_2) = U(T(x_1,x_2)) := u(x_1,x_2)\,. \tag{31}$$

Since $u \in H^1_0(Q_+)$, we have $U \in H^1_0\big(\Omega_{(1-\varepsilon,1)}\big)$. Clearly, U satisfies condition (26) (because it is axisymmetric by definition); moreover, using (30) and (31), it is readily checked that U satisfies also conditions (27) and (28). Hence $U \in \widetilde{H}\big(\Omega_{(1-\varepsilon,1)}\big)$, and therefore

$$\widetilde{\lambda}_1\big(\Omega_{(1-\varepsilon,1)}\big) \le \frac{\int_{\Omega_{(1-\varepsilon,1)}} |\nabla U|^2}{\int_{\Omega_{(1-\varepsilon,1)}} |U|^2}\,. \tag{32}$$

Let $\mathcal{T}_l$ and $\mathcal{T}_u$ denote the lower and upper triangles

$$\mathcal{T}_l := Q_+ \cap \{x_1 \ge 0,\ x_2 \ge 0,\ x_1+x_2 \le 1\} \qquad \text{and} \qquad \mathcal{T}_u := Q_+ \cap \{x_1 \ge 0,\ x_2 \ge 0,\ x_1+x_2 \ge 1\}\,.$$

Some straightforward computations yield

$$\begin{aligned}\int_{\Omega_{(1-\varepsilon,1)}} |U|^2(X_1,X_2)\,dX_1\,dX_2 &= 4\Big\{\int_{\mathcal{T}_l} |u|^2(x_1,x_2)\,dx_1dx_2 + (1-\varepsilon)\int_{\mathcal{T}_u} |u|^2(x_1,x_2)\,dx_1dx_2\Big\} \\ &= 4\Big\{\frac{1}{4}+\varepsilon\Big(\frac{1}{\pi^2}-\frac{1}{8}\Big)\Big\}\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_{(1-\varepsilon,1)}} |\nabla U|^2(X_1,X_2)\,dX_1\,dX_2 &= 4\Big\{\int_{\mathcal{T}_l} |\nabla u|^2(x_1,x_2)\,dx_1dx_2 \\ &\quad +(1-\varepsilon)\int_{\mathcal{T}_u}\Big[\Big(\frac{\partial u}{\partial x_1}\frac{1}{1-\varepsilon}\Big)^2 + \Big(\frac{\partial u}{\partial x_1}\frac{\varepsilon}{1-\varepsilon}+\frac{\partial u}{\partial x_2}\Big)^2\Big]\,dx_1dx_2\Big\} \\ &= 4\Big\{\frac{\pi^2}{16}+\frac{\varepsilon}{4}+\frac{\pi^2}{16}\Big(\frac{1-\varepsilon+\varepsilon^2}{1-\varepsilon}\Big)\Big\}\end{aligned}$$

Inserting these expressions into (32) gives the desired estimate (29). □

Proposition 16. *Let $P := (1,1)$, and set*

$$\mathcal{U}(P) := \Big\{(x_1,x_2) \in \Delta \ : \ x_1 + x_2 \geq 2 - \overline{\varepsilon}\Big\}$$

where Δ is the triangular region defined in (20)*, and $\overline{\varepsilon} := 0.371034$. If $\Omega_{(x_1,x_2)}$ is the octagon defined in* (21)*, it holds*

$$\mathcal{E}(\Omega_{x_1,x_2}) = \lambda_1\big(\Omega_{(x_1,x_2)}\big)\big|\Omega_{(x_1,x_2)}\big| < 2\pi^2 \qquad \forall (x_1,x_2) \in \mathcal{U}(P)\,. \tag{33}$$

Proof. As a first step, let us show that

$$\widetilde{\lambda}_1\big(\Omega_{(1-\varepsilon,1)}\big)\big|\Omega_{(1-\varepsilon,1)}\big| < 2\pi^2 \qquad \forall \varepsilon \in (0,\overline{\varepsilon}]\,. \tag{34}$$

Namely, from Lemma 15, and taking into account that $\big|\Omega_{(1-\varepsilon,1)}\big| = 4 - 2\varepsilon$, we infer that the inequality (34) is fulfilled for all $\varepsilon \in (0,1)$ such that the function

$$\varphi(\varepsilon) := \frac{\frac{\pi^2}{16} + \frac{\varepsilon}{4} + \frac{\pi^2}{16}\Big(\frac{1-\varepsilon+\varepsilon^2}{1-\varepsilon}\Big)}{\frac{1}{4} + \varepsilon\Big(\frac{1}{\pi^2} - \frac{1}{8}\Big)}\big(4 - 2\varepsilon\big) - 2\pi^2$$

is nonpositive. Since the smallest zero of φ is easily computed to be

$$-\frac{4 + 2\pi^2 - \sqrt{144 - 16\pi^2 + 4\pi^4}}{8 - 2\pi^2} = 0.371034 = \overline{\varepsilon}\,,$$

we deduce that (34) holds.
The remaining of the proof consists in showing that (34) implies (33).
Notice first that it is equivalent to state the inequality (33) only for $(x_1,x_2) \in \mathcal{U}(P) \cap \{x_2 \geq x_1\}$, since octagons corresponding to $(x_1,x_2) \in \mathcal{U}(P) \cap \{x_2 \leq x_1\}$ are obtained by a rotation from octagons corresponding to $(x_1,x_2) \in \mathcal{U}(P) \cap \{x_2 \geq x_1\}$.
Next we observe that, by definition of $\mathcal{U}(P)$, for every $(x_1,x_2) \in \mathcal{U}(P) \cap \{x_2 \geq x_1\}$, the point $(x_1 + x_2 - 1, 1)$ is of the form $(1-\varepsilon, 1)$ for some $\varepsilon \in (0,\overline{\varepsilon}]$. Taking also into account that the octagon $\Omega_{(x_1,x_2)}$ and the hexagon $\Omega_{(x_1+x_2-1,1)}$ have the same area, and in view of Remark 14, we infer that the required implication (34) $\Rightarrow$ (33) is satisfied if

$$\widetilde{\lambda}_1\big(\Omega_{(x_1,x_2)}\big) \leq \widetilde{\lambda}_1\big(\Omega_{(x_1+x_2-1,1)}\big) \qquad \forall (x_1,x_2) \in \Delta \cap \{x_2 \geq x_1\}\,. \tag{35}$$

In order to prove (35), we exploit the tool of continuous Steiner (de-)symmetrization due to Brock (*cf.* Figure 2 below).

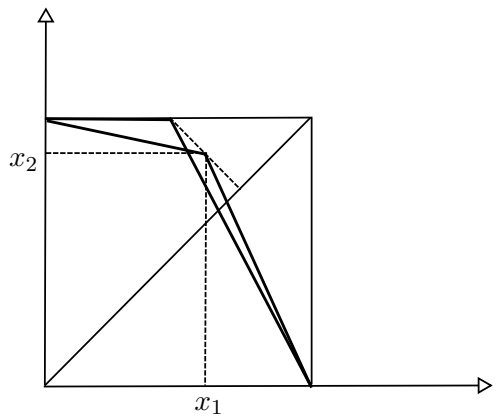

FIGURE 2. De-symmetrization of the set $\Omega_{(x_1,x_2)} \cap (\mathbb{R}_+)^2$

Let $(x_1,x_2)\in\Delta\cap\{x_2\geq x_1\}$ be fixed, and let $u\in\widetilde{H}\big(\Omega_{(x_1+x_2-1,1)}\big)$ be a function at which the infimum which defines $\widetilde{\lambda}_1\big(\Omega_{(x_1+x_2-1,1)}\big)$ is attained. For convenience, after extending it to zero outside $\Omega_{(x_1+x_2-1,1)}$, we may think of u as a function defined on $\mathbb{R}^2$. Then, we consider the restriction of u to $(\mathbb{R}_+)^2$, and we extend it to a function $\widehat{u}$, defined on the whole of $\mathbb{R}^2$, in such a way that, on the complement of $(\mathbb{R}_+)^2$, $\widehat{u}$ is symmetric with respect to the line $x_1=x_2$. Note that this is possible because $u\in\widetilde{H}\big(\Omega_{(x_1+x_2-1,1)}\big)$, so that in particular it satisfies condition (27).
By construction we have

$$\widetilde{\lambda}_1\big(\Omega_{(x_1+x_2-1,1)}\big)=\frac{\int_{\Omega_{(x_1+x_2-1,1)}}|\nabla u|^2}{\int_{\Omega_{(x_1+x_2-1,1)}}|u|^2}=\frac{\int_{(\mathbb{R}_+)^2}|\nabla u|^2}{\int_{(\mathbb{R}_+)^2}|u|^2}=\frac{\int_{(\mathbb{R}_+)^2}|\nabla\widehat{u}|^2}{\int_{(\mathbb{R}_+)^2}|\widehat{u}|^2}\,.$$

Now we write

$$\frac{\int_{(\mathbb{R}_+)^2}|\nabla\widehat{u}|^2}{\int_{(\mathbb{R}_+)^2}\widehat{u}^2}=\frac{\int_{\mathbb{R}^2}|\nabla\widehat{u}|^2-\int_{\mathbb{R}^2\setminus(\mathbb{R}_+)^2}|\nabla\widehat{u}|^2}{\int_{\mathbb{R}^2}|\widehat{u}|^2-\int_{\mathbb{R}^2\setminus(\mathbb{R}_+)^2}|\widehat{u}|^2}\,.$$

Let $\widehat{u}_t$ be a continuous Steiner symmetrization of $\widehat{u}$ with respect to the line $x_1=x_2$. Recall that $\widehat{u}_t$ is defined by the equalities

$$\{\widehat{u}_t>c\}=\{\widehat{u}>c\}^t\quad\forall c>0\qquad\text{and}\qquad\{\widehat{u}_t=0\}=\mathbb{R}^2\setminus\bigcup_{c>0}\{\widehat{u}>c\}^t\,,$$

where $\{\widehat{u}>c\}^t$ denotes a continuous Steiner symmetrization of the level set $\{\widehat{u}>c\}$ with respect to the line $x_1=x_2$. For the benefit of the reader who is not familiar with this kind of symmetrization, let us also specify that, for any measurable set $M\subset\mathbb{R}^2$, a continuous Steiner symmetrization of M with respect to a given direction ν is the set

$$M^t:=\{x\nu+y\nu^\perp\ :\ x\in\mathbb{R}\,,\ y\in M(x)^t\}\,,$$

where $M(x):=(x\nu+\mathbb{R}\nu^\perp)\cap M$, and $M(x)^t:=\Phi_t(M(x))$, $\{\Phi_t\}_{t\geq0}$ being a family of transformations from the class of measurable subsets of $\mathbb{R}$ into itself, such that: (i) $\Phi_t(M)$ are equimeasurable (ii) Φ_t preserve inclusions (iii) Φ_t have the semigroup property (iv) if I is an interval of the form $[a,b]$, it holds $\Phi_t(I)=[a_t,b_t]$, with $a_t:=\frac12(a-b+e^{-t}(a+b))$, $b_t:=\frac12(b-a+e^{-t}(a+b))$. For more details on the definition and properties of continuous Steiner symmetrization, we refer to [13, 14].
We observe that, for all $t>0$, $\widehat{u}_t$ turns out to agree with $\widehat{u}$ on $\mathbb{R}^2\setminus(\mathbb{R}_+)^2$. This follows by combining the fact that $\widehat{u}$ is symmetric with respect to the line $x_1=x_2$ on the set $\mathbb{R}^2\setminus(\mathbb{R}_+)^2$ with the fact that u satisfies conditions (27) and (28).
We deduce that, for all $t>0$, there holds:

$$\int_{\mathbb{R}^2\setminus(\mathbb{R}_+)^2}|\nabla\widehat{u}|^2=\int_{\mathbb{R}^2\setminus(\mathbb{R}_+)^2}|\nabla\widehat{u}_t|^2\qquad\text{and}\qquad\int_{\mathbb{R}^2\setminus(\mathbb{R}_+)^2}|\widehat{u}|^2=\int_{\mathbb{R}^2\setminus(\mathbb{R}_+)^2}|\widehat{u}_t|^2\,.$$

On the other hand, by the properties of continuous Steiner symmetrization, we have:

$$\int_{\mathbb{R}^2}|\nabla\widehat{u}|^2\geq\int_{\mathbb{R}^2}|\nabla\widehat{u}_t|^2\qquad\text{and}\qquad\int_{\mathbb{R}^2}|\widehat{u}|^2=\int_{\mathbb{R}^2}|\widehat{u}_t|^2$$

(see *e.g.* [14, Corollary 3.2 and eq. (2.31)]):
So far, we have obtained

$$\widetilde{\lambda}_1\big(\Omega_{(x_1+x_2-1,1)}\big)\geq\frac{\int_{\mathbb{R}^2}|\nabla\widehat{u}_t|^2-\int_{\mathbb{R}^2\setminus(\mathbb{R}_+)^2}|\nabla\widehat{u}_t|^2}{\int_{\mathbb{R}^2}|\widehat{u}_t|^2-\int_{\mathbb{R}^2\setminus(\mathbb{R}_+)^2}|\widehat{u}_t|^2}=\frac{\int_{(\mathbb{R}_+)^2}|\nabla\widehat{u}_t|^2}{\int_{(\mathbb{R}_+)^2}|\widehat{u}_t|^2}\tag{36}$$

Now, we denote by $\widehat{v}_t$ the axisymmetric function on $\mathbb{R}^2$ which agrees with $\widehat{u}_t$ on $(\mathbb{R}_+)^2$. Clearly, we have

$$\frac{\int_{(\mathbb{R}_+)^2} |\nabla \widehat{u}_t|^2}{\int_{(\mathbb{R}_+)^2} |\widehat{u}_t|^2} = \frac{\int_{\mathbb{R}^2} |\nabla \widehat{v}_t|^2}{\int_{\mathbb{R}^2} |\widehat{v}_t|^2} \,. \tag{37}$$

Finally we observe that, when t varies in $[0, +\infty)$, the set $\{\widehat{v}_t > 0\}$ is of the form $\Omega_{(x,-x+x_1+x_2)}$, with x ranging in $[x_1 + x_2 - 1, \frac{x_1+x_2}{2})$. Therefore, by continuity, there exists $\bar{t} \in (0, +\infty)$ such that that $\{\widehat{v}_{\bar{t}} > 0\} = \Omega_{(x_1,x_2)}$. For such $\bar{t}$, we have by construction $\widehat{v}_{\bar{t}} \in \widetilde{H}(\Omega_{(x_1,x_2)})$, and hence

$$\frac{\int_{\mathbb{R}^2} |\nabla \widehat{v}_{\bar{t}}|^2}{\int_{\mathbb{R}^2} |\widehat{v}_{\bar{t}}|^2} \geq \widetilde{\lambda}_1(\Omega_{(x_1,x_2)}) \,. \tag{38}$$

By combining (36), (37) and (38), we obtain the required inequality (35), and our proof is achieved. □

5. Confidence zone near the segment S

Recall that, according to definitions (23) and (25), the segment S is a portion of the boundary of the square Q_- (actually, $S = \partial Q_- \cap (\mathbb{R}_+)^2$). In this section we prove that the function $\mathcal{E}(x_1, x_2) = \lambda_1(\Omega_{(x_1,x_2)})|\Omega_{(x_1,x_2)}|$ assumes values smaller than $2\pi^2$ in a computable neighborhood of S. Since this is a delicate task, let us outline our strategy, and explain how the confidence zone will be obtained.

- As an initial step, in Section 5.1 we perform the shape derivative of the functional $\mathcal{E}$ with respect to a deformation field with transforms the square Q_- into an octagon. Its expression can be explicitly computed (thanks to the knowledge of the first eigenfunction of Q_-), and turns out to be strictly negative, except at the endpoints of S where it tends to 0 (see Proposition 17). The negative sign ensures the existence, at every point of S except its endpoints, of a small segment orthogonal to S where $\mathcal{E} < 2\pi^2$; the matter is precisely to find an explicit estimate for the length of such a segment. On the other hand, the vanishing of the shape derivative at the extremities is a warning that finding a confidence zone near $(1, 0)$ where $\mathcal{E} < 2\pi^2$ will require more efforts, since in a neighborhood of such point a second order effect comes into play.
- In Section 5.2 we find a computable confidence region whose thickness, as expected, degenerates when approaching $(0, 1)$ (see Proposition 18). This is obtained by taking a test function defined as a suitable deformation of the first eigenfunction of Q_-. The shape of this confidence region is represented in Figure 4 right; notice that its thickness is bounded below in the complement of any fixed ball centered at $(0, 1)$, and therefore this region will do the job outside such a ball. We are thus reduced to find a confidence region near $(0, 1)$. This task is achieved in two steps as described hereafter.
- In Section 5.3, we determine a neighborhood of the point $(0, 1)$ where the inequality $\mathcal{E} < 2\pi^2$ holds true (see Proposition 19). This is obtained by using a clever cut-off argument from [1], combined with a careful balance of angles and radii. The shape of the resulting confidence region is represented in Figure 5.

- To summarize, the *confidence zone near* S is built as follows:

  (i) Choose a neighborhood of $(0,1)$ as in Proposition 19 to determine a first confidence zone.

  (ii) Take the intersection of the region given by Proposition 18 with the complement of the neighborhood determined in Proposition 19 to determine a second confidence zone.

  (iii) Take the union of the zones found at items (i) and (ii).

5.1. **The shape derivative.** It will be repeatedly useful to recall that the first Dirichlet eigenfunction of Q_-, normalized so that $\int_{Q_-} u^2 = 1$ is given by

$$u(x,y) = \frac{\cos(\pi x) + \cos(\pi y)}{\sqrt{2}} \qquad \forall (x,y) \in Q_- \tag{39}$$

(see for instance [26, Theorem 2.5.1]).
We are going to compute the shape derivative of $\mathcal{E}$ at the square Q_-, with respect to a deformation field which transforms it into an octagon. More precisely, for $\varepsilon > 0$, we consider the one-parameter family of deformations of Q_- given by

$$Q_\varepsilon := (\mathrm{Id} + \varepsilon V)(Q_-)\,, \tag{40}$$

where the velocity field V is defined as follows: it is symmetric with respect to the coordinate axes and in the first quadrant it is given by

$$V(x,y) = V_n(x,y)n\,, \qquad V_n\big(x,y\big) := \begin{cases} a\dfrac{x}{x_0} & \text{if } x \in [0,x_0] \\[2ex] a\,\dfrac{y}{1-x_0} & \text{if } x \in [x_0,1]\,, \end{cases} \tag{41}$$

where $n = \frac{(1,1)}{\sqrt{2}}$ is the unit outer normal to ∂Q_-, x_0 is a fixed point in $(0,1)$, and a is a positive parameter ($a = V_n\big(x_0, 1-x_0\big)$, see Figure 3).

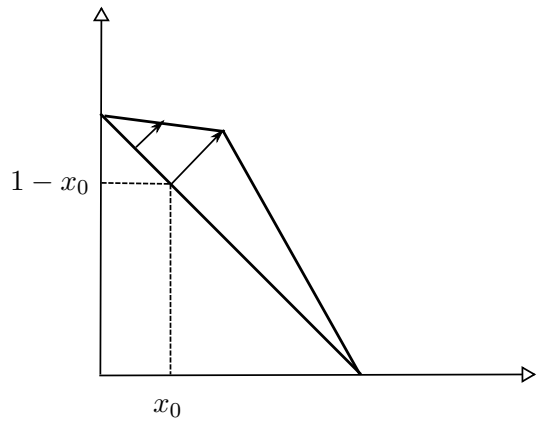

FIGURE 3. The deformation field in Proposition 17

Proposition 17. *For $\varepsilon > 0$, let Q_ε be defined as in (40), with V as in (41). Then the shape derivative*

$$\frac{d}{d\varepsilon}\Big(\lambda_1(Q_\varepsilon)|Q_\varepsilon|\Big)\Big|_{\varepsilon=0^+}$$

is strictly negative (and tends to 0 as $x_0 \to 0$ or $x_0 \to 1$).

Proof. We have

$$\frac{d}{d\varepsilon}\Big(\lambda_1(Q_\varepsilon)|Q_\varepsilon|\Big)\Big|_{\varepsilon=0^+} = \lambda_1(Q_-)\frac{d}{d\varepsilon}|Q_\varepsilon|\Big|_{\varepsilon=0^+} + |Q_-|\frac{d}{d\varepsilon}\Big(\lambda_1(Q_\varepsilon)\Big)\Big|_{\varepsilon=0^+}, \tag{42}$$

with (see for instance [26, Chapter 2])

$$\frac{d}{d\varepsilon}|Q_\varepsilon|\Big|_{\varepsilon=0^+} = \int_{\partial Q} V_n \, d\mathcal{H}^1 \tag{43}$$

$$\frac{d}{d\varepsilon}\Big(\lambda_1(Q_\varepsilon)\Big)\Big|_{\varepsilon=0^+} = -\int_{\partial Q} V_n |\nabla u|^2 \, d\mathcal{H}^1 , \tag{44}$$

u being the first eigenfunction of Q_- normalized so that $\int_{Q_-} u^2 = 1$.
The r.h.s. of (43) is immediately computed as

$$\begin{aligned}\int_{\partial Q_-} V_n \, d\mathcal{H}^1 &= 4\int_S V_n \, d\mathcal{H}^1 \\ &= 4\Big[\int_0^{x_0}\Big(a\frac{t}{x_0}\Big)\sqrt{2}\,dt + \int_{x_0}^1\Big(a\frac{1-t}{1-x_0}\Big)\sqrt{2}\,dt\Big] \\ &= 2\sqrt{2}\,a\end{aligned}$$

In order to compute the r.h.s. of (44), we exploit the knowledge of the explicit expression of u according to (39). In particular, the evaluation of $|\nabla u|^2$ on S gives

$$\big|\nabla u(x, 1-x)\big|^2 = \pi^2 \sin^2\big(\pi x\big) \qquad \forall x \in \big[0,1\big] .$$

Therefore,

$$\begin{aligned}\int_{\partial Q_-} |\nabla u|^2 V_n \, d\mathcal{H}^1 &= 4\int_S |\nabla u|^2 V_n \, d\mathcal{H}^1 \\ &= 4\Big[\int_0^{x_0}\big[\pi^2\sin^2\big(\pi t\big)\big]\Big(a\frac{t}{x_0}\Big)\sqrt{2}\,dt \\ &\quad + \int_{x_0}^1\big[\pi^2\sin^2\big(\pi t\big)\big]\Big(a\frac{1-t}{1-x_0}\Big)\sqrt{2}\,dt\Big] \\ &= 4\pi^2 a\sqrt{2}\Big\{\frac{1}{x_0}\int_0^{x_0}\sin^2\big(\pi t\big)\,t\,dt + \frac{1}{1-x_0}\int_{x_0}^1 \sin^2\big(\pi t\big)\,(1-t)\,dt \\ &= \pi^2 a\sqrt{2}\Big[1 + \frac{\sin^2(\pi x_0)}{\pi^2 x_0(1-x_0)}\Big] .\end{aligned}$$

Inserting these expressions and the equalities $|Q_-| = 2$, $\lambda_1(Q_-) = \pi^2$ into (42), we get

$$\frac{d}{d\varepsilon}\Big(\lambda_1(Q_\varepsilon)|Q_\varepsilon|\Big)\Big|_{\varepsilon=0^+} = -2a\sqrt{2}\,\frac{\sin^2(\pi x_0)}{x_0(1-x_0)} .$$

which yields the statement. □

5.2. **The confidence region arbitrarily close to the extremities of S.**

Proposition 18. *Let $\theta \in (0, \frac{\pi}{4})$ be a fixed angle. There exists a computable value $\overline{h} = \overline{h}(\theta) > 0$ such that, for all points (a, b) of the form $(\frac{1}{1+\tan(\theta)} + h\cos\theta, \frac{\tan(\theta)}{1+\tan(\theta)} + h\sin\theta)$ (see Figure 4, left), it holds*

$$\lambda_1\big(\Omega_{(a,b)}\big)\big|\Omega_{(a,b)}\big| < 2\pi^2 \qquad \text{if } h \in (0, \overline{h})\,. \tag{45}$$

In particular, inequality (45) is satisfied for (a, b) belonging to the region represented in grey in Figure 4, right.

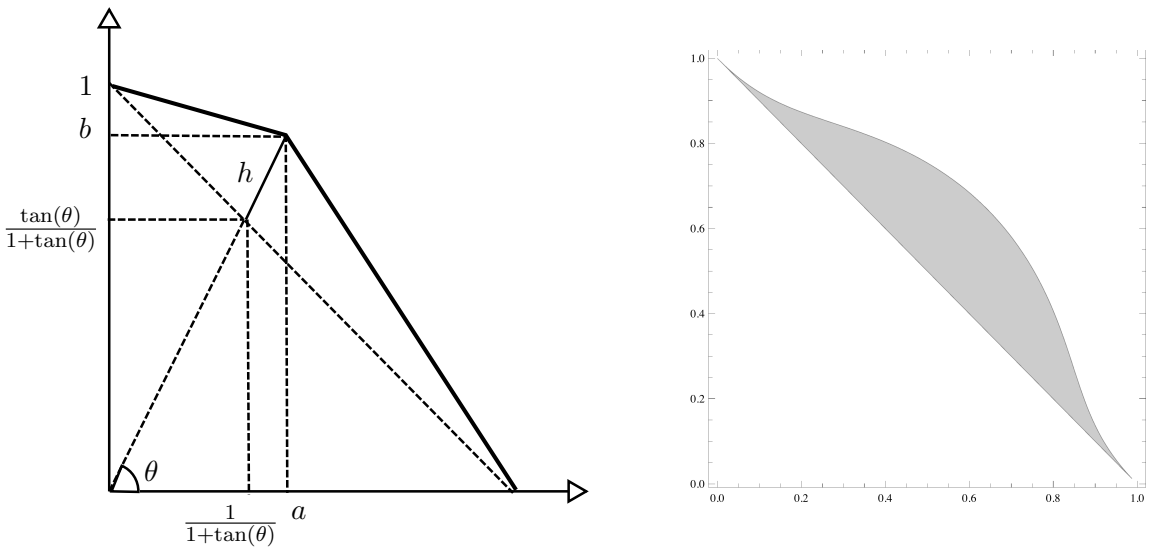

FIGURE 4. Geometry of Proposition 18 and related confidence region

Proof. To prove the statement, we are going to construct a test function $U \in H^1_0(\Omega_{(a,b)})$ whose Rayleigh quotient multiplied by the area of $\Omega_{(a,b)}$ satisfies

$$\Psi(a, b) := \frac{\int_{\Omega_{(a,b)}} |\nabla U|^2}{\int_{\Omega_{(a,b)}} |U|^2}\big|\Omega_{(a,b)}\big| < 2\pi^2\,. \tag{46}$$

for all points (a, b) as in the statement, *i.e.*, of the form $(\frac{1}{1+\tan(\theta)} + h\cos\theta, \frac{\tan(\theta)}{1+\tan(\theta)} + h\sin\theta)$, with $h \in (0, \overline{h})$.

Notice that when writing (a, b) in such a form, the parameter h represents the distance between (a, b) and the intersection point of the straight lines $y = 1 - x$ and $y = \tan(\theta)x$ (see Figure 4 left).

We define U as the axisymmetric function given on $\Omega^+_{(a,b)} := \Omega_{(a,b)} \cap \{x_1 \geq 0, x_2 \geq 0\}$ by

$$U(X_1, X_2) := \begin{cases} u(T'(x_1, x_2)) & \text{if } (X_1, X_2) \in R' := \Omega^+_{(a,b)} \cap \{x_2 \geq \tan(\theta)x_1\} \\ \\ u(T''(x_1, x_2)) & \text{if } (X_1, X_2) \in R'' := \Omega^+_{(a,b)} \cap \{x_2 \leq \tan(\theta)x_1\}\,, \end{cases}$$

where u is the first Dirichlet eigenfunction of Q_- according to (39), and T', T'' are the affine transformations defined respectively by

$$T'(x_1,x_2)=\Big(\frac{a}{l}x_1,\frac{b-(1-l)}{l}x_1+x_2\Big)\qquad\text{and}\qquad T''(x_1,x_2)=\Big(x_1+\frac{a-l}{1-l}x_2,\frac{b}{1-l}x_2\Big).$$

Here we have introduced for brevity the parameter

$$l=l(\theta):=\frac{1}{1+\tan(\theta)}\,,$$

so that the straight lines $y=1-x$ and $y=\tan(\theta)x$ meet at $(l,1-l)$.
Notice that T' and T'' are uniquely determined by the conditions

$$\begin{cases}T'(0,0)=(0,0)\\ T'(0,1)=(0,1)\\ T'(l,1-l)=(a,b)\end{cases}\qquad\text{and}\qquad\begin{cases}T''(0,0)=(0,0)\\ T''(1,0)=(1,0)\\ T''(l,1-l)=(a,b)\,.\end{cases}$$

Let us compute the Rayleigh quotient of U as a function of h. We have

$$\int_{\Omega(a,b)}U^2=4\int_{\Omega^+(a,b)}U^2=4\int_{R'}U^2+4\int_{R''}U^2=f_1(h)+f_2(h)\,,$$

with

$$f_1(h):=\frac{2a}{l}\int_0^l dx_1\int_{x_1\tan(\theta)}^{1-x_1}\big[\cos(\pi x_1)+\cos(\pi x_2)\big]^2\,dx_2$$

$$f_2(h):=\frac{2b}{1-l}\int_0^{1-l}dx_2\int_{x_2\cot(\theta)}^{1-x_2}\big[\cos(\pi x_1)+\cos(\pi x_2)\big]^2\,dx_1\,.$$

Clearly the integrals which appear in the above equations do not depend on h and can be easily computed in terms of θ. Taking also into account that a and b are affine functions of h, we see that $f_1(h)$ and $f_2(h)$ are first order polynomials in h (with coefficients depending on θ). Their explicit expressions read:

$$\begin{aligned}f_1(h)\;&=(4\pi^2)^{-1}\Big[5\cos\Big(\tfrac{2\pi}{\tan(\theta)+1}\Big)+\cos\Big(\tfrac{2\pi\tan(\theta)}{\tan(\theta)+1}\Big)\\&+\tfrac{32\cos^2\left(\frac{\pi}{\tan(\theta)+1}\right)}{\cot(\theta)-1}-2\cot(\theta)\sin^2\Big(\tfrac{\pi\tan(\theta)}{\tan(\theta)+1}\Big)\\&+2\sin^2\Big(\tfrac{\pi}{\tan(\theta)+1}\Big)\big(5\tan(\theta)+\sec^2(\theta)\big)+4\pi^2-6\Big]\Big(h\cos(\theta)+\tfrac{1}{\tan(\theta)+1}\Big)\end{aligned}$$

and

$$\begin{aligned}f_2(h)\;&=(16\pi^2(\tan(\theta)-1))^{-1}\Big[32(\tan(\theta)+1)\cos\Big(\tfrac{2\pi}{\cot(\theta)+1}\Big)\\&+2\csc^2(\theta)\sec^2(\theta)\Big(2\left(\pi^2-2\right)\sin(2\theta)-\pi^2\sin(4\theta)\\&+\cos(4\theta)\Big(\cos\Big(\tfrac{2\pi}{\tan(\theta)+1}\Big)+\pi^2-7\Big)\\&+(2\sin(4\theta)+1)\cos\Big(\tfrac{2\pi}{\tan(\theta)+1}\Big)-\pi^2+5\Big)\Big]\Big(h\sin(\theta)+\tfrac{\tan(\theta)}{\tan(\theta)+1}\Big)\,.\end{aligned}$$

We now turn to the computation of the integral of $|\nabla U|^2$. We exploit the identities

$$\begin{aligned}|\nabla U|^2\;&=\Big(\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial X_1}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial X_1}\Big)^2+\Big(\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial X_2}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial X_2}\Big)^2\\&=\Big|\frac{\partial(X_1,X_2)}{\partial(x_1,x_2)}\Big|^{-2}\Big[\Big(\frac{\partial u}{\partial x_1}\frac{\partial X_2}{\partial x_2}-\frac{\partial u}{\partial x_2}\frac{\partial X_2}{\partial x_1}\Big)^2+\Big(-\frac{\partial u}{\partial x_1}\frac{\partial X_1}{\partial x_2}+\frac{\partial u}{\partial x_2}\frac{\partial X_1}{\partial x_1}\Big)^2\Big]\end{aligned}$$

So we obtain

$$\int_{\Omega(a,b)} |\nabla U|^2 = 4\int_{\Omega^+(a,b)} |\nabla U|^2 = 4\int_{R'} |\nabla U|^2 + 4\int_{R''} |\nabla U|^2 = g_1(h) + g_2(h)$$

with

$$g_1(h) := \frac{2\pi^2 l}{a}\int_0^l dx_1 \int_{x_1\tan(\theta)}^{1-x_1} \Big[\Big(-\sin(\pi x_1) + \sin(\pi x_2)\frac{b-(1-l)}{l}\Big)^2 + \Big(\sin(\pi x_2)\frac{a}{l}\Big)^2\Big]\, dx_2$$

and

$$g_2(h) := \frac{2\pi^2(1-l)}{b}\int_0^{1-l} dx_2 \int_{x_2\cot(\theta)}^{1-x_2} \Big[\Big(\sin(\pi x_1)\frac{b}{1-l}\Big)^2 + \Big(\sin(\pi x_1)\frac{a-l}{1-l} - \sin(\pi x_2)\Big)^2\Big]\, dx_1\,.$$

Recalling again that a and b are affine in h, we see that g_1 and g_2 are rational functions of h, and precisely that each of them is the quotient of a second order polynomial in h by a first order polynomial in h. By computing explicitly the integrals above, we get the following explicit expressions:

$$\begin{aligned}
g_1(h) \;&= \Big[4\Big(h\cos(\theta) + \tfrac{1}{\tan(\theta)+1}\Big)\Big]^{-1}\Big\{ -\tfrac{h^2\tan(\theta)(\csc(\theta)+\sec(\theta))^2}{2(\tan(\theta)+1)^2}\Big[-2\left(1+\pi^2\right)\sin(2\theta)\\
&+(\sin(2t)-\cos(2\theta)+1)\cos\Big(\tfrac{2\pi}{\tan(\theta)+1}\Big) + (\sin(2\theta)+\cos(2\theta)+1)\cos\Big(\tfrac{2\pi\tan(\theta)}{\tan(\theta)+1}\Big) - 2\Big]\\
&+4h\Big[-2\sin(\theta)\sin^2\Big(\tfrac{\pi}{\tan(\theta)+1}\Big) - 2\sin(\theta)\cos^2(\theta)\sec(2\theta)\Big(\cos\Big(\tfrac{2\pi}{\tan(\theta)+1}\Big)+3\Big)\\
&+\cos(\theta)\Big(\tfrac{\pi^2}{\tan(\theta)+1} + \cot(\theta)\sin^2\Big(\tfrac{\pi\tan(\theta)}{\tan(\theta)+1}\Big) + (2\sec(2\theta)-1)\sin^2\Big(\tfrac{\pi}{\tan(\theta)+1}\Big)\Big)\Big]\\
&+\tfrac{(\tan(\theta)+1)\cot(\theta)}{(\cot(\theta)+1)^3}\Big[\pi^2\csc^2(\theta) - 2\cot(\theta)\cos^2\Big(\tfrac{\pi}{\tan(\theta)+1}\Big) + (2\cot(\theta)+1)\cos\Big(\tfrac{2\pi}{\tan(\theta)+1}\Big)\\
&+\cot^2(\theta)\Big(-\cos\Big(\tfrac{2\pi\tan(\theta)}{\tan(\theta)+1}\Big) + 3\pi^2+1\Big) + 2\cot^3(\theta)\sin^2\Big(\tfrac{\pi\tan(\theta)}{\tan(\theta)+1}\Big) - \pi^2 - 1\Big]\Big\}
\end{aligned}$$

and

$$\begin{aligned}
g_2(h) \;&= \Big[4(h\sin(\theta)+h\cos(\theta)+1)\Big]^{-1}\Big\{2h^2(\cot(\theta)+1)\Big[-\cos\Big(\tfrac{2\pi\cot(\theta)}{\cot(\theta)+1}\Big)\\
&+\csc(\theta)\sec(\theta)\sin^2\Big(\tfrac{\pi\cot(\theta)}{\cot(\theta)+1}\Big) + \pi^2+1\Big]\\
&+\tfrac{4h\sin(\theta)(\cot(\theta)+1)^2}{(\tan(\theta)+1)^2}\Big[\pi^2\tan^2(\theta) + \tan(\theta)\left(\tan^2(\theta)+\tan(\theta)-1\right)\sin^2\Big(\tfrac{\pi}{\tan(\theta)+1}\Big)\\
&+\cos\Big(\tfrac{2\pi}{\tan(\theta)+1}\Big) + 24\sin^4(\theta)\csc(4\theta) + \tan^2(\theta)\sin^2\Big(\tfrac{\pi\cot(\theta)}{\cot(\theta)+1}\Big)\\
&-2(\tan(\theta)+1)\tan(2\theta)\sin^2\Big(\tfrac{\pi}{\cot(\theta)+1}\Big)\\
&+\sec(2\theta)\Big(2\sin^2(\theta)(\tan(\theta)+1)\cos\Big(\tfrac{2\pi}{\cot(\theta)+1}\Big)+3\Big) - 4\Big]\\
&-\frac{\csc(\theta)(\csc(\theta)+\sec(\theta))\Big(2\cos(2\theta)\sin^2\Big(\frac{\pi}{\tan(\theta)+1}\Big)+\cot(\theta)\Big(\cos(2\theta)\Big(-\cos\Big(\frac{2\pi}{\tan(\theta)+1}\Big)+2\pi^2+1\Big)-2\pi^2\Big)\Big)}{(\cot(\theta)+1)^2}\Big\}\,.
\end{aligned}$$

Finally, the area of $|\Omega_{(a,b)}|$ is readily computed as a first order polynomial in h, as

$$|\Omega_{a,b}|(h) = 2\Big[1 + h(\cos(\theta)+\sin(\theta))\Big]\,.$$

By using the above expressions for $f_1(h)$, $f_2(h)$, $g_1(h)$, $g_2(h)$, and $|\Omega_{a,b}|(h)$, we are enabled to write explicitly the quantity $\Psi(a,b)$ at the left hand side of (46) as a function of h, and precisely as the quotient between a polynomial of degree 4 by a polynomial of degree 3.

Therefore, inequality (46) holds true for $h \in (0, \overline{h})$, where $\overline{h}$ is the first strictly positive zero of a fourth order polynomial in h (vanishing at $h = 0$). By taking the union of the segments $(0, \overline{h})$ for varying θ, we obtain that inequality (45) is satisfied in the region plotted (by the use of Mathematica) in Figure 4 right. □

5.3. **The confidence region near** $(0, 1)$**.**

Proposition 19. *For points* (a, b) *of the form* $\big(\rho\cos\alpha, 1 - \rho\sin\alpha\big)$*, the inequality*

$$\lambda_1\big(\Omega_{(a,b)}\big)\big|\Omega_{(a,b)}\big| < 2\pi^2$$

holds true if either $\rho \in (0, 0.06]$ *and* $\alpha \in \big[0, \frac{\pi}{5}\big)$*, or* $\rho \in (0, 0.1]$ *and* $\alpha \in \big[\frac{\pi}{5}, \frac{\pi}{4}\big)$ *(see Figure 5).*

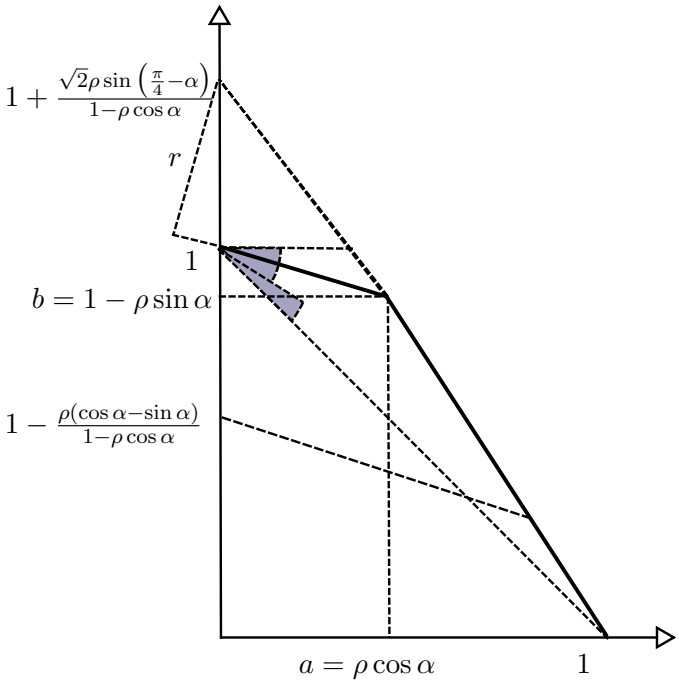

FIGURE 5. Geometry of Proposition 19 and related confidence region (in grey)

Proof. We need to introduce some notation. For brevity, throughout the proof we denote by $\Omega_{\rho,\alpha}$ the octagon $\Omega_{(\rho\cos\alpha, 1-\rho\sin\alpha)}$, and by $T_{\rho,\alpha}$ the rhombus (containing $\Omega_{\rho,\alpha}$) whose intersection with the first quadrant is the triangle with vertices $(0, 0)$, $(0, 1)$ and $(0, 1+\delta)$, with

$$\delta = \delta(\rho, \alpha) := \frac{\rho(\cos\alpha - \sin\alpha)}{1 - \rho\cos\alpha} = \frac{\sqrt{2}\rho\sin\left(\frac{\pi}{4} - \alpha\right)}{1 - \rho\cos\alpha}\,.$$

We set

$$A_+ := \Omega_{\rho,\alpha} \cap \big\{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \ : \ y \le -(\tan\alpha)x + 1 - \delta\big\}$$

$$B_+ := \Omega_{\rho,\alpha} \cap \big\{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \ : \ -(\tan\alpha)x + 1 - \delta \le y \le -(\tan\alpha)x + 1\big\}$$

$$C_+ := T_{\rho,\alpha} \cap \big\{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \ : \ y \ge -(\tan\alpha)x + 1\big\}\,,$$

see Figure 6 below.

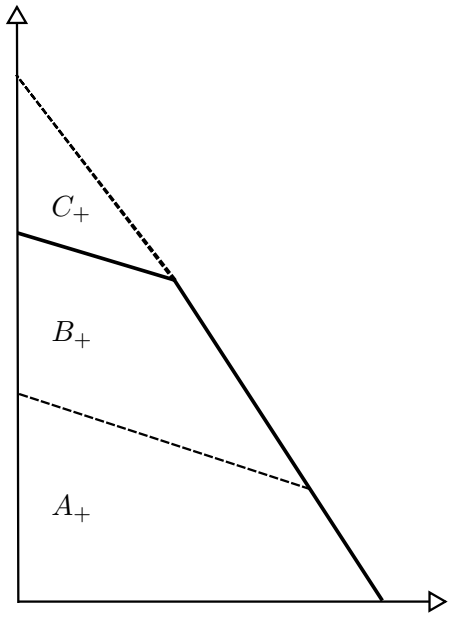

FIGURE 6. The regions A_+, B_+, and C_+.

By construction, it holds

$$\Omega_{\rho,\alpha} \cap (\mathbb{R}_+ \times \mathbb{R}_+) = \Big(T_{\rho,\alpha} \cap (\mathbb{R}_+ \times \mathbb{R}_+)\Big) \setminus C_+ = A_+ \cup B_+\,.$$

We set A, B, C, the axisymmetric domains such that

$$A_+ = A \cap (\mathbb{R}_+ \times \mathbb{R}_+)\,, \qquad B_+ = B \cap (\mathbb{R}_+ \times \mathbb{R}_+)\,, \qquad C_+ = C \cap (\mathbb{R}_+ \times \mathbb{R}_+)\,.$$

We define $u \in H^1_0(T_{\rho,\alpha})$ as the test function obtained by computing the first eigenfunction of the square Q_- at $(x, \frac{y}{1+\delta})$, namely

$$u(x,y) := \frac{1}{\sqrt{2}}\Big[\cos(\pi x) + \cos\big(\frac{\pi y}{1+\delta}\big)\Big]\,,$$

and we set

$$M := \max_{B\cup C} u$$

$$f := \int_{T_{\rho,\alpha}} u^2 = f(\delta) = (1+\delta)$$

$$g := \int_{T_{\rho,\alpha}} |\nabla u|^2 = g(\delta) = \frac{\pi^2}{2}\frac{1+(1+\delta)^2}{1+\delta}\,.$$

We finally let

$$r := \delta\cos\alpha = \frac{\sqrt{2}\rho\sin\big(\frac{\pi}{4}-\alpha\big)}{1-\rho\cos\alpha}\cos\alpha\,,$$

and we divide the remaining of the proof in four steps.

Step 1: We claim that the inequality $\lambda_1(\Omega_{\rho,\alpha})|\Omega_{\rho,\alpha}| < 2\pi^2$ holds true provided the following three estimates are satisfied

$$M^2 \leq \frac{f}{8|C|} \tag{47}$$

$$\frac{M^2}{r^2} \leq \frac{4g}{|T_{\rho,\alpha}|} \tag{48}$$

$$\frac{2M^2}{r^2} + 16\pi^2 M^2 + 4\pi^2\delta \leq (4-\gamma)\pi^2\,, \quad \text{with} \quad \gamma \geq 2\frac{1-\tan\alpha}{1-\rho\sin\alpha}\,. \tag{49}$$

In order to bound from above $\lambda_1(\Omega_{\rho,\alpha})$ we consider the solution η to the following boundary value problem in an infinite strip

$$\begin{cases} \Delta\eta = 0 & \text{in } \{-(\tan\alpha)x + 1 - \delta \leq y \leq -(\tan\alpha)x + 1\} \\ \eta = M & \text{on } \{y = -(\tan\alpha)x + 1 - \delta\} \\ \eta = 0 & \text{on } \{y = -(\tan\alpha)x + 1\}\,. \end{cases}$$

Clearly η depends only on the distance d from the straight line $\{y = -(\tan\alpha)x + 1 - \delta\}$, and it is immediate to compute it as $\eta(d) = -\frac{M}{r}d + M$. In particular, we observe for later use that

$$\eta'(r) = -\frac{M}{r}\,. \tag{50}$$

We consider the axisymmetric function w given in the first quadrant by

$$w := \begin{cases} u & \text{in } A_+ \\ u \wedge \eta := \min\{u, \eta\} & \text{in } B_+\,. \end{cases}$$

Since $w \in H^1_0(\Omega_{\rho,\alpha})$, it can be chosen as a test function in the Rayleigh quotient of $\lambda_1(\Omega_{\rho,\alpha})$:

$$\lambda_1(\Omega_{\rho,\alpha})|\Omega_{\rho,\alpha}| \leq \frac{\int_{A\cup B}|\nabla w|^2}{\int_{A\cup B} w^2}|\Omega_{\rho,\alpha}|\,. \tag{51}$$

We now estimate separately the integrals of w^2 and of $|\nabla w|^2$.
Recalling the definitions of f and M given above, we have:

$$\int_{A\cup B} w^2 = f + \int_B ((u\wedge\eta)^2 - u^2) - \int_C u^2 \geq f - M^2|C| - M^2|B| = f\Big(1 - \frac{4M^2|C|}{f}\Big)\,,$$

where the last equality follows from the geometric relation $|B| = 3|C|$. Hence, by exploiting the elementary inequality $\frac{1}{1-t} \leq 1 + 2t$ holding for $t \in [0, \frac{1}{2}]$, together with the assumed condition (47), we obtain

$$\frac{1}{\int_{A\cup B} w^2} \leq \frac{1}{f}\Big(1 + \frac{8M^2|C|}{f}\Big)\,. \tag{52}$$

We now turn to the integral of $|\nabla w|^2$. Similarly as above, recalling the definition of g, we have:

$$\int_{A\cup B}|\nabla w|^2 = g + \int_B (|\nabla(u\wedge\eta)|^2 - |\nabla u|^2) - \int_C |\nabla u|^2$$

By applying the elementary inequality $|v_1|^2 - |v_2|^2 \le 2v_1 \cdot (v_1 - v_2)$, we get

$$\int_B (|\nabla(u\wedge\eta)|^2 - |\nabla u|^2) = \int_{B\cap\{u>\eta\}} (|\nabla(u\wedge\eta)|^2 - |\nabla u|^2) \le 2\int_{B\cap\{u>\eta\}} \nabla(u\wedge\eta)\cdot\nabla(u\wedge\eta - u)\,.$$

We now integrate by parts. Taking into accout that η is harmonic in B, that the only portion of $\partial\big(B\cap\{u>\eta\}\big)$ where $u-\eta\neq 0$ is the line segment $L := \{y = -\tan(\theta)x + 1\}\cap\Omega_{\rho,\alpha}$, and recalling (50), we obtain

$$\begin{aligned} 2\int_B \nabla(u\wedge\eta)\cdot\nabla(u\wedge\eta - u) &= 2\int_{B\cap\{u>\eta\}} \nabla\eta\cdot\nabla(\eta - u) \\ &= 2\int_L \frac{\partial\eta}{\partial\nu}(-u) = \frac{2M}{r}\int_L u\,. \end{aligned}$$

An integration on straight lines perpendicular to L yields

$$\int_L u \le \int_C |\nabla u|\,.$$

Summarizing, we have

$$\int_{A\cup B} |\nabla w|^2 \le g + \frac{2M}{r}\int_C |\nabla u| - \int_C |\nabla u|^2 \le g + \frac{M^2|C|}{r^2}\,, \tag{53}$$

where the last inequality follows by applying first the elementary inequality $-a^2 - b^2 \le -2ab$ and then the Cauchy-Schwarz inequality.
In view of (51), (52), and (53), we have

$$\begin{aligned} \lambda_1(\Omega_{\rho,\alpha})|\Omega_{\rho,\alpha}| &\le \frac{1}{f}\Big(g + \frac{M^2|C|}{r^2}\Big)\Big(|T_{\rho,\alpha}| - 4|C|\Big)\Big(1 + \frac{8M^2|C|}{f}\Big) \\ &\le \Big[\frac{g|T_{\rho,\alpha}|}{f} - |C|\Big(\frac{4g}{f} - \frac{M^2}{r^2}\frac{|T_{\rho,\alpha}|}{f}\Big)\Big]\Big(1 + \frac{8M^2|C|}{f}\Big) \\ &\le \frac{g|T_{\rho,\alpha}|}{f} - |C|\Big(\frac{4g}{f} - \frac{M^2}{r^2}\frac{|T_{\rho,\alpha}|}{f} - 8M^2 g\frac{|T_{\rho,\alpha}|}{f^2}\Big)\,, \end{aligned}$$

where the last inequality holds true thanks to the assumed condition (48).
By using the expressions of f, g, and $|T_{\rho,\alpha}|$ in terms of δ, the above inequality can be rewritten as

$$\begin{aligned} \lambda_1(\Omega_{\rho,\alpha})|\Omega_{\rho,\alpha}| &\le 2\pi^2 + \frac{\pi^2\delta^2}{1+\delta} - |C|\Big(4\pi^2 - \frac{2M^2}{r^2} - \frac{8M^2\pi^2\big(1 + (1+\delta)^2\big)}{(1+\delta)^2} - \frac{2\pi^2(2\delta + \delta^2)}{(1+\delta)^2}\Big) \\ &< 2\pi^2 + \frac{\pi^2\delta^2}{1+\delta} - |C|\Big(4\pi^2 - \frac{2M^2}{r^2} - 16M^2\pi^2 - 4\pi^2\delta\Big) \\ &\le 2\pi^2 + \frac{\pi^2\delta^2}{1+\delta} - \pi^2\gamma\delta\rho\frac{\cos\alpha}{2}\,. \end{aligned}$$

where in the last inequality we have exploited the assumed condition (49) and the identity $|C| = \frac{\delta\rho\cos\alpha}{2}$. Finally by the choice of γ, we have

$$\gamma \ge \frac{2}{\rho\cos\alpha}\frac{\delta}{1+\delta} = 2\frac{1-\tan\alpha}{1-\rho\sin\alpha}\,,$$

which concludes the proof of the claim in Step 1.

Step 2: We claim that

$$\frac{M}{r\rho} \le \frac{\pi^2}{(1-\rho^*)\sin\left(\frac{\pi}{4}-\alpha^*\right)} \qquad \text{if } \rho \in (0,\rho^*] \text{ and } \alpha \in [0,\alpha^*] \tag{54}$$

$$\frac{M}{r\rho} \le 2\pi^2(1+\delta)\Big[1+\frac{1}{1-\rho^*}\Big]\cos\alpha^* \qquad \text{if } \rho \in (0,\rho^*] \text{ and } \alpha \in \big[\alpha^*, \tfrac{\pi}{4}\big)\,. \tag{55}$$

Inequality (54) is obtained as follows:

$$\begin{aligned}
\frac{M}{r} &\le \frac{u\Big(0, \frac{1+\delta-2(\delta+\rho\sin\alpha)}{1+\delta}\Big)}{\frac{\sqrt{2}\rho\cos\alpha\sin\left(\frac{\pi}{4}-\alpha\right)}{1-\rho\cos\alpha}} = \frac{1}{2}\frac{1+\cos\Big[\pi\Big(\frac{1+\delta-2(\delta+\rho\sin\alpha)}{1+\delta}\Big)\Big]}{\frac{\rho\cos\alpha\sin\left(\frac{\pi}{4}-\alpha\right)}{1-\rho\cos\alpha}} \\
&\le \frac{1}{2\sin\left(\frac{\pi}{4}-\alpha^*\right)}\frac{1-\cos\Big[\pi\Big(\frac{2(\delta+\rho\sin\alpha)}{1+\delta}\Big)\Big]}{\frac{\rho\cos\alpha}{1-\rho\cos\alpha}} \\
&= \frac{1}{2\sin\left(\frac{\pi}{4}-\alpha^*\right)}\frac{1-\rho\cos\alpha}{\rho\cos\alpha}2\sin^2\Big[\pi\Big(\frac{\delta+\rho\sin\alpha}{1+\delta}\Big)\Big] \\
&\le \frac{1}{\sin\left(\frac{\pi}{4}-\alpha^*\right)}\frac{1-\rho\cos\alpha}{\rho\cos\alpha}\pi^2\Big(\frac{\delta+\rho\sin\alpha}{1+\delta}\Big)^2 \\
&= \frac{\pi^2}{\sin\left(\frac{\pi}{4}-\alpha^*\right)}\frac{\rho\cos\alpha\,(1-\rho\sin\alpha)^2}{(1+\delta)^2(1-\rho\cos\alpha)} \\
&\le \frac{\pi^2\rho}{(1-\rho^*)\sin\left(\frac{\pi}{4}-\alpha^*\right)}\,.
\end{aligned}$$

Inequality (55) is obtained as follows. Denote by

$$\nu := \frac{1}{\sqrt{1+(1+\delta)^2}}\big(1+\delta, 1\big)$$

the unit outer normal to $T_{\rho,\alpha}$ along its side on the line $y=(1+\delta)(1-x)$. Then

$$\begin{aligned}
\frac{M}{2r} &\le \max_{B\cup C}|\nabla u\cdot\nu| \\
&= \frac{\pi}{\sqrt{2}}\frac{1+\delta}{\sqrt{1+(1+\delta)^2}}\max_{B\cup C}\Big|\sin(\pi x)+\frac{1}{(1+\delta)^2}\sin\Big(\frac{\pi y}{1+\delta}\Big)\Big| \\
&= \frac{\pi}{\sqrt{2}}\frac{1+\delta}{\sqrt{1+(1+\delta)^2}}\Big[\sin(2\pi\rho\cos\alpha)+\frac{1}{(1+\delta)^2}\sin\Big(\frac{2\pi(\delta+\rho\sin\alpha)}{1+\delta}\Big)\Big] \\
&\le \frac{\pi}{\sqrt{2}}\frac{1+\delta}{\sqrt{1+(1+\delta)^2}}\Big[2\pi\rho\cos\alpha+\frac{2\pi(\delta+\rho\sin\alpha)}{(1+\delta)^3}\Big] \\
&= \frac{2\pi^2}{\sqrt{2}}\frac{1+\delta}{\sqrt{1+(1+\delta)^2}}\Big[\rho\cos\alpha+\frac{1}{(1+\delta)^3}\frac{\rho\cos\alpha(1-\rho\sin\alpha)}{1-\rho\cos\alpha}\Big] \\
&\le \pi^2(1+\delta)\rho\Big[1+\frac{1}{1-\rho^*}\Big]\cos\alpha^*\,.
\end{aligned}$$

We point out that the first inequality above is obtained by estimating u along the edge $(B_+ \cup C_+) \cap \{(x,y) : x = 0\}$ via Lagrange Theorem.

Step 3: We claim that the estimates (47)-(48)-(49) *are satisfied for* $\rho \le 0.1$ *and* $\alpha \in [\frac{\pi}{5}, \frac{\pi}{4}]$.

With the help of (55), let us firstly check that (49) is satisfied with $\gamma = 0.608$. Such choice of γ is motivated by the inequality

$$0.608 \ge 2\frac{1-\tan\left(\frac{\pi}{5}\right)}{0.9} \ge 2\frac{1-\tan\alpha}{1-\rho\sin\alpha} \qquad \forall \rho \in [0, 0.1],\ \forall \alpha \in \left[\frac{\pi}{5}, \frac{\pi}{4}\right].$$

For such value of γ, inequality (49) reads

$$\frac{2M^2}{r^2} + 16\pi^2 M^2 + 4\pi^2\delta \le 3.392\,\pi^2\,. \tag{56}$$

We observe that, for all $\rho \in [0, 0.1]$ and $\alpha \in \left[\frac{\pi}{5}, \frac{\pi}{4}\right]$, there holds

$$\delta = \frac{\sqrt{2}\rho\sin\left(\frac{\pi}{4}-\alpha\right)}{1-\rho\cos\alpha} \le \sqrt{2}\sin\left(\frac{\pi}{20}\right)\frac{0.1}{0.9} \le 0.0246 \tag{57}$$

and

$$r = \delta\cos\alpha \le \sqrt{2}\sin\left(\frac{\pi}{20}\right)\cos\left(\frac{\pi}{5}\right)\frac{0.1}{0.9} \le 0.0199\,. \tag{58}$$

By using (57), (58), and the estimate (55) (with $\rho^* = 0.1$ and $\alpha^* = \frac{\pi}{5}$), we obtain:

$$4\pi^2\delta \le 0.099\,\pi^2$$

$$16\pi^2 M^2 \le \pi^2\Big[64\pi^4 r^2\rho^2(1+\delta)^2\Big(1+\frac{1}{0.9}\Big)^2\cos^2\left(\frac{\pi}{5}\right)\Big] \le 0.757\,\pi^2$$

$$\frac{2M^2}{r^2} \le \pi^2\Big[8\pi^2\rho^2(1+\delta)^2\Big(1+\frac{1}{0.9}\Big)^2\cos^2\left(\frac{\pi}{5}\right)\Big] \le 2.418\,\pi^2$$

and eventually

$$\frac{2M^2}{r^2} + 16\pi^2 M^2 + 4\pi^2\delta \le (2.418 + 0.757 + 0.099)\pi^2 < 3.392\,\pi^2\,,$$

which ensures (56) and hence (49). The inequalities obtained above for the terms $16\pi^2 M^2$ and $\frac{2M^2}{r^2}$ readily imply that also the inequalities (47) and (48) are satisfied.

Step 4: We claim that the estimates (47)-(48)-(49) *are satisfied for* $\rho \le 0.06$ *and* $\alpha \in [\frac{\pi}{12}, \frac{\pi}{4}]$.

We proceed in a similar way as above. Since

$$2.128 \ge 2\frac{1}{1-0.06} \ge 2\frac{1-\tan\alpha}{1-\rho\sin\alpha} \qquad \forall \rho \in [0, 0.06],\ \forall \alpha \in \left[\frac{\pi}{12}, \frac{\pi}{4}\right],$$

let us check that (49) is satisfied with $\gamma = 2.128$. For such value of γ, inequality (49) reads

$$\frac{2M^2}{r^2} + 16\pi^2 M^2 + 4\pi^2\delta \le 1.872\,\pi^2\,. \tag{59}$$

We observe that, for all $\rho \in [0, 0.06]$ and $\alpha \in \left[\frac{\pi}{12}, \frac{\pi}{4}\right]$, there holds

$$\delta = \frac{\sqrt{2}\rho\sin\left(\frac{\pi}{4}-\alpha\right)}{1-\rho\cos\alpha} \le \sqrt{2}\sin\left(\frac{\pi}{6}\right)\frac{0.06}{0.94} \le 0.046 \tag{60}$$

and

$$r = \delta \cos\alpha \le \sqrt{2}\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{12}\right)\frac{0.06}{0.94} \le 0.044\,. \tag{61}$$

By using (60), (61), and the estimate (55) (with $\rho^* = 0.06$ and $\alpha^* = \frac{\pi}{12}$), we obtain:

$$4\pi^2\delta \le 0.184\,\pi^2$$

$$16\pi^2 M^2 \le \pi^2\Big[64\pi^4 r^2\rho^2(1+\delta)^2\Big(1+\frac{1}{0.94}\Big)^2\cos^2\left(\frac{\pi}{12}\right)\Big] \le 0.189\,\pi^2$$

$$\frac{2M^2}{r^2} \le \pi^2\Big[8\pi^2\rho^2(1+\delta)^2\Big(1+\frac{1}{0.94}\Big)^2\cos^2\left(\frac{\pi}{12}\right)\Big] \le 1.236\,\pi^2$$

and eventually

$$\frac{2M^2}{r^2} + 16\pi^2 M^2 + 4\pi^2\delta \le (1.236 + 0.189 + 0.184)\pi^2 < 1.872\,\pi^2\,,$$

which ensures (59) and hence (49). Also in this case, the inequalities obtained above for the terms $16\pi^2 M^2$ and $\frac{2M^2}{r^2}$ readily imply that the weaker inequalities (47) and (48) are satisfied.

Step 5: We claim that the estimates (47)-(48)-(49) *are satisfied for* $\rho \le 0.06$ *and* $\alpha \in [0, \frac{\pi}{12}]$. We still have

$$2.128 \ge 2\frac{1}{1-0.06} \ge 2\frac{1-\tan\alpha}{1-\rho\sin\alpha} \qquad \forall\rho \in (0, 0.06],\ \forall\alpha \in \Big[0, \frac{\pi}{12}\Big]\,,$$

so we need to check that, as in Step 4, the inequality (59) is satisfied. Now, for $\rho \in (0, 0.06]$ and $\alpha \in \big[0, \frac{\pi}{12}\big]$, there holds

$$\delta = \frac{\sqrt{2}\rho\sin\left(\frac{\pi}{4}-\alpha\right)}{1-\rho\cos\alpha} \le \sqrt{2}\sin\left(\frac{\pi}{4}\right)\frac{0.06}{0.94} \le 0.06383 \tag{62}$$

and

$$r = \delta\cos\alpha \le \delta \le 0.06383\,. \tag{63}$$

By using (62), (63), and the estimate (54) (with $\rho^* = 0.06$ and $\alpha^* = \frac{\pi}{12}$), we obtain:

$$4\pi^2\delta \le 0.256\,\pi^2$$

$$16\pi^2 M^2 \le \pi^2\Big[16\pi^4 r^2\rho^2\big(0.94\big)^{-2}\sin^{-2}\left(\frac{\pi}{6}\right)\Big] \le 0.1035\,\pi^2$$

$$\frac{2M^2}{r^2} \le \pi^2\Big[2\pi^2\rho^2\big(0.94\big)^{-2}\sin^{-2}\left(\frac{\pi}{6}\right)\Big] \le 0.3217\,\pi^2$$

and eventually

$$\frac{2M^2}{r^2} + 16\pi^2 M^2 + 4\pi^2\delta \le (0.3217 + 0.1035 + 0.256)\pi^2 < 1.872\,\pi^2\,,$$

which ensures (59) and hence (49). As usual, also the weaker estimates (47) and (48) are satisfied. □

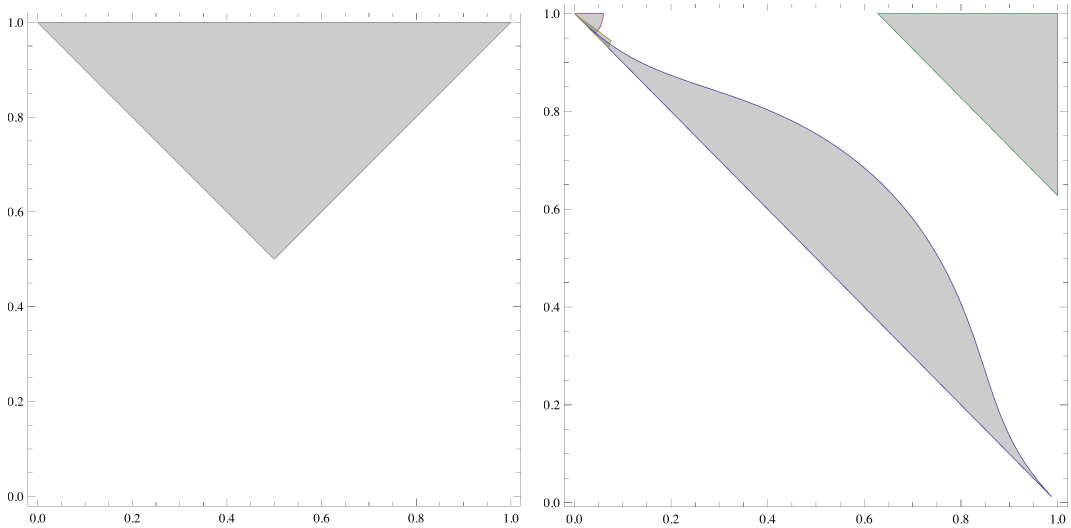

FIGURE 7. Analysis region on the left, analytically identified confidence zone on the right.

6. Numerical computations outside the confidence zones

For symmetry reasons, it is enough to consider only the octagons $\Omega_{(x,y)}$, for (x, y) belonging to the region in Figure 7, left. Following the analytical results obtained in Sections 4 and 5, the confidence zone has a geometry which is displayed in Figure 7, right. The remaining, non-confidence region, is covered with computational squares of different sizes, *e.g.* $ABCD$.

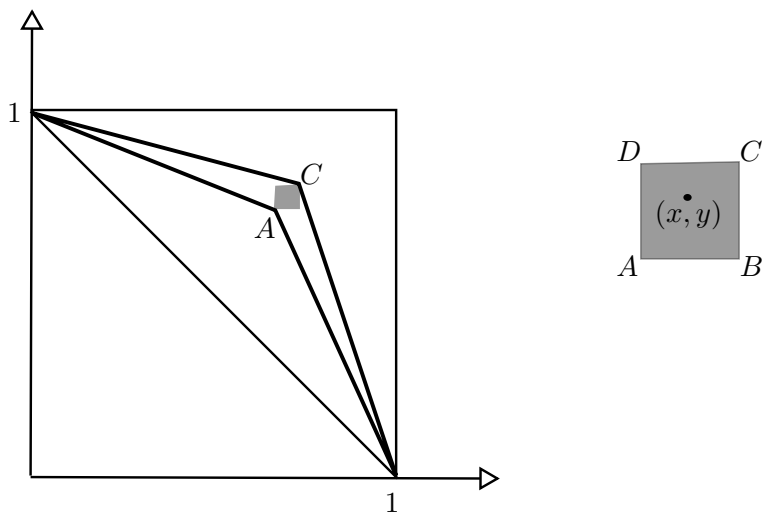

FIGURE 8. Clearing the square ABCD.

In order to prove that the analytic inequality holds true for every $\Omega_{(x,y)}$ with $(x, y) \in ABCD$, we use the following chain of inequalities

$$\lambda_1(\Omega_{(x,y)})|\Omega_{(x,y)}| \leq \lambda_1(\Omega_A)|\Omega_C| \leq \lambda_1^{num}(\Omega_A)|\Omega_C|.$$

The first inequality holds true since the Dirichlet eigenvalue is decreasing for domain inclusions, while the measure is increasing. In the second inequality, $\lambda_1^{num}(\Omega_A)$ stands for the numerically computed value. This inequality is also true, as a consequence of the numerical method itself. We use finite elements approximation for $\lambda_1(\Omega_A)$ with a triangular mesh which fits precisely with the octagon Ω_A, with no boundary approximation. Since the finite elements functional space P_1 of affine functions is strictly contained in the Sobolev space $H_0^1(\Omega_A)$, as a consequence of the Rayleigh quotient formulation for the first eigenvalue, the numerically computed value $\lambda_1^{num}(\Omega_A)$ will not be smaller than the analytical value $\lambda_1(\Omega_A)$. The numerically computed value is expected to be "exact", in the sense that the numerical computation produces an effective piecewise affine function which formally can be taken as test function in the Rayleigh quotient for the octagon, and the computation of the Rayleigh quotient associated to this function is exact.
Finally, it is enough to find a suitable covering of the non-confidence zone by computational squares $ABCD$, and to prove that for every such a square the following inequality holds

$$0 < 2\pi^2 - \lambda_1^{num}(\Omega_A)|\Omega_C|. \tag{64}$$

As one can easily notice, larger the size of the rectangle is, the deviation of $\lambda_1^{num}(\Omega_A)|\Omega_C|$ from the product $\lambda_1(\Omega_{(x,y)})|\Omega_{(x,y)}|$ is higher, leading to a possible value above $2\pi^2$.
From a practical point of view, we cover the complement of the confidence zone with square blocks of size between 10^{-2} and 10^{-1}. Each of these blocks is divided in 1 to 10^4 smaller computational squares $ABCD$, depending on how far away it lays from the critical region. The confidence region being very narrow near the upper vertex of the segment S, the size of the computational squares is 10^{-4} (see Figures 9 and 10).
The heaviest computation occurred for the square block with a lower-left vertex in $(0.06, 0.95)$ and size 10^{-2}, which was divided in 10^4 computational squares and required 1 hour of computations on a 1.8 GHz computer with 4 Go Ram using MATLAB [1] and the Partial Differential Equation Toolbox. On the opposite, the square block with lower-left corner at $(0.6, 0.7)$ and size 10^{-1} was divided in 25 computational squares and required 10 seconds of computation. Computations are done with double precision (15 to 17 exact digits) and the rule to decide that inequality (64) holds true is that the first non zero digit in the right hand side occurs at most at the order 10^{-4}.
In Figure 10, we notice the presence of three triangular regions T_1, T_2, T_3 outside the confidence zone and not covered by any block of size 10^{-2}. They are treated individually, each one being covered by a small square with minimal size in order to minimize the computation time.
All the results of the computations are available at `http://www.lama.univ-savoie.fr/~bucur/computations/`.

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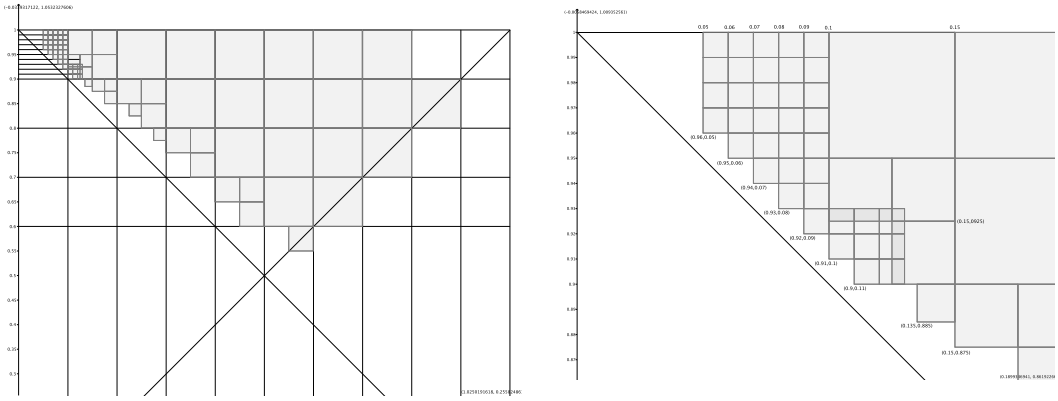

FIGURE 9. The computation blocks covering the non-confidence region, and a zoom near $(0,1)$.

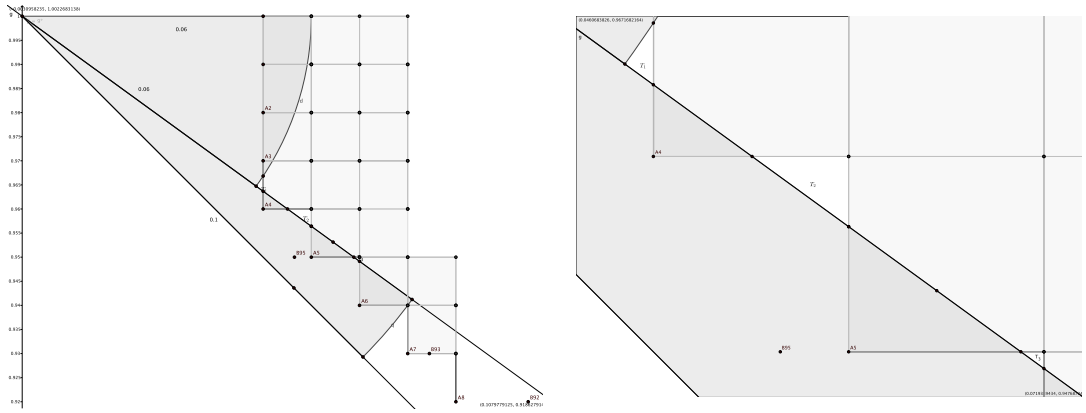

FIGURE 10. Contact with the confidence region near $(0,1)$ and a zoom at the interface.

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