Weak semi-continuity of the duality product in Sobolev spaces

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Abstract

Given a weakly convergent sequence of positive functions in $W_0^{1,p}(\Omega)$, we prove the equivalence between its convergence in the sense of obstacles and the lower semicontinuity of the term by term duality product associated to (the p-Laplacian of) weakly convergent sequences of p-superharmonic functions of $W_0^{1,p}(\Omega)$. This result implicitly gives new characterizations for both the convergence in the sense of obstacles of a weakly convergent sequence of positive functions and for the weak l.s.c of the duality product.

Keywords: duality product, γ-convergence, obstacle convergence

1 Introduction

The limit of the scalar product of two weakly convergence sequences in a Hilbert space is, a priori, uncontrollable. In some particular situations, as for example in Sobolev spaces, when the sequences of functions are solutions (or supersolutions) of partial differential equations, extra information can be obtained on the scalar product of the limits by using qualitative properties of the solutions of the PDEs. In this paper, we are interested in duality products in the Sobolev spaces $W^{-1,q} \times W_0^{1,p}$ involving p-superharmonic and positive functions. We characterize all sequences of positive functions, such that the duality product with the p-Laplacian of p-superharmonic functions is lower semincontinuous.

More precisely, let $(u_n), (v_n) \subseteq W_0^{1,p}(\Omega)$ be two sequences of non-negative functions and assume they weakly converge to u and v, respectively. Assuming moreover that $-\Delta_p u_n \geq 0$ in the sense of distributions on Ω, we wonder whether

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla v_n dx \geq \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx. \quad (1)$$

Under no further assumption, this assertion is in general false. For an extensive study of this question we refer the reader to [3] (see also [4] for further results), where the authors give sufficient conditions on the sequence (v_n) in order that (1) holds true. Precisely, the main hypothesis is that the functions v_n are also p-superharmonic $-\Delta_p v_n \geq 0$. An example showing that the positivity condition $v_n \geq 0$ is not (in general) sufficient for the lower semicontinuity of (1) is also given for $p = 2$. The example relies on the emergence of the "strange term" appearing in the relaxation process of domains through γ-convergence (see [10]) and gives an intuitive hint on the fact that, in order that (1) is true, (v_n) should vary such that their level sets do not produce relaxation measures via γ-convergence (see [5] for a detailed introduction to γ-convergence and Section 2 for a short review).

The purpose of this paper is to give a characterization of all $W_0^{1,p}(\Omega)$-weakly convergent sequences (v_n) of nonnegative functions for which (1) holds true for every $W_0^{1,p}(\Omega)$-weakly convergent sequence (u_n) such that $-\Delta_p u_n \geq 0$.

We prove that the necessary and sufficient condition that (v_n) has to satisfy is to converge in the sense of obstacles (see Section 2 for the precise definition and [1, 12] for details). This convergence is, in a certain sense, weaker than the strong convergence of $W_0^{1,p}(\Omega)$ and stronger than the weak convergence of $W_0^{1,p}(\Omega)$. Since (v_n) is assumed by hypothesis weakly convergent in $W_0^{1,p}(\Omega)$, proving that it also converges in the sense of obstacles is equivalent to the possibility of finding a sequence θ_n strongly convergent to v in $W_0^{1,p}(\Omega)$, such that $\theta_n \leq v_n$ a.e. This is a consequence of the characterization of the obstacle convergence via the Mosco convergence of the convex sets

$$K_{v_n} = \{u \in W_0^{1,p}(\Omega) : u \leq v_n \text{ a.e.}\}.$$

In order to describe the obstacles, we make use on fine quasi-continuity properties of Sobolev functions. The proof of the main result of the paper relies on the characterization of the obstacle convergence $v_n \xrightarrow{\text{obst}} v$ in terms of the γ-convergence of the level sets $\{v_n > t\} \xrightarrow{\gamma} \{v > t\}$, and on the knowledge of the relaxed measures associated to a γ-convergent sequence of quasi-open sets.

Of course, the diffcult part is the necessity. In [3], the hypothesis on the p-superharmonicity of v_n insures the fact that $\min\{v_n, v\}$ converges strongly to v in $W_0^{1,p}(\Omega)$! So, taking $\theta_n = \min\{v_n, v\}$ we recover the obstacle convergence and fall into the sufficient part of the characterization result.

All results in this paper hold for A-superharmonic functions, where $-\text{div } A$ is a non-linear operator of p-Laplacian type. Precisely, assuming $A : W_0^{1,p}(\Omega) \mapsto W^{-1,q}(\Omega)$ is similar to the p-Laplacian (see the exact definition in Section 2) one can prove that if $(v_n) \subseteq W_0^{1,p}(\Omega)$ is a weakly convergent sequence of non-negative functions, then v_n converges in the sense of obstacles to the same limit v if and only if for every sequence of functions $(u_n) \subseteq W_0^{1,p}(\Omega)$, such that $-\text{div } (a(x, \nabla u_n)) \geq 0$ and $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ we have

$$\liminf_{n\to\infty} \int_\Omega a(x, \nabla u_n) \nabla v_n dx \geq \int_\Omega a(x, \nabla u) \nabla v dx. \tag{2}$$

For the simplicity of the exposition, our results are presented for the p-Laplace operator. We point out the fact that the convergence of (v_n) into the sense of obstacles is independent on the choice of the operator $-\text{div } (a(x, \cdot))$.

Section 2 contains a review of the main tools used in the paper, Section 3 contains the proof of the characterization result and the last section is devoted to some examples. A particular attention is given to uniformly oscillating obstacles.

2 Obstacles, capacity and γ-convergence

Capacity and relaxation measures. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and let $1 < p < +\infty$. The p-capacity of a subset E in Ω is

$$\mathrm{cap}_p(E, \Omega) = \inf\Big\{ \int_\Omega |\nabla u|^p \, dx \, : u \in U_E \Big\} ,$$

where U_E is the set of all functions u of the Sobolev space $W_0^{1,p}(\Omega)$ such that $u \geq 1$ almost everywhere in a neighborhood of E.

If a property $P(x)$ holds for all $x \in E$ except for the elements of a set $Z \subseteq E$ with $\mathrm{cap}_p(Z) = 0$, we say that $P(x)$ holds p-quasi-everywhere on E and write p-q.e. The expression *almost everywhere* refers, as usual, to the Lebesgue measure.

A subset A of Ω is said to be *p-quasi-open* if for every $\epsilon > 0$ there exists an open subset A_ϵ of Ω, such that $A \subseteq A_\epsilon$ and $\mathrm{cap}_p(A_\epsilon \setminus A, \Omega) < \epsilon$. A function $f \colon \Omega \to \mathbb{R}$ is said to be *p-quasi-continuous* (resp. *quasi-lower semi-continuous*) if for every $\epsilon > 0$ there exists a continuous (resp. lower semi-continuous) function $f_\epsilon \colon \Omega \to \mathbb{R}$ such that $\mathrm{cap}_p(\{f \neq f_\epsilon\}, \Omega) < \epsilon$, where $\{f \neq f_\epsilon\} = \{x \in \Omega : f(x) \neq f_\epsilon(x)\}$. It is well known (see, e.g., [17, 19]) that every function u of the Sobolev space $W_0^{1,p}(\Omega)$ has a p-quasi-continuous representative, which is uniquely defined up to a set of p-capacity zero. We shall always identify the function u with its quasi-continuous representative, so that a point-wise condition can be imposed on $u(x)$ for p-quasi-every $x \in \Omega$.

Since p is fixed throughout the paper, the index p may be dropped when speaking about p-quasi-open sets, p-quasi-continuity, etc.

We denote by $\mathcal{M}_0^p(\Omega)$ the set of all nonnegative Borel measures μ on Ω, such that

i) $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\mathrm{cap}_p(B, \Omega) = 0$

ii) $\mu(B) = \inf\{\mu(U) : U \; p-\text{quasi-open}, \; B \subseteq U\}$ for every Borel set $B \subseteq \Omega$.

We stress the fact that the measures $\mu \in \mathcal{M}_0^p(\Omega)$ do not need to be finite, and may take the value $+\infty$.

There is a natural way to identify a quasi-open set to a measure. More generally, given an arbitrary Borel subset $E \subseteq \Omega$, we denote by $\infty|_E$ the measure defined by

i) $\infty|_E(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\mathrm{cap}_p(B \cap E, \Omega) = 0$,

ii) $\infty|_E(B) = +\infty$ for every Borel set $B \subseteq \Omega$ with $\mathrm{cap}_p(B \cap E, \Omega) > 0$.

Definition 2.1 *We say that a sequence (μ_n) of measures in $\mathcal{M}_0^p(\Omega)$ γ_p-converges to a measure $\mu \in M_0^p(\Omega)$ if and only if $F_{\mu_n} : W_0^{1,p}(\Omega) \to \overline{\mathbb{R}}$,*

$$F_{\mu_n}(u) = \int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p d\mu_n$$

Γ-converges in $L^p(\Omega)$ to F_μ, where

$$F_\mu(u) = \int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p d\mu.$$

In order to simplify notations and since p is fixed, we drop the index p and instead of γ_p we note γ.

We recall that $F_n : W_0^{1,p}(\Omega) \to \overline{\mathbb{R}}$ Γ-converges to F in $L^p(\Omega)$ if for every $u \in L^p(\Omega)$ there exists a sequence $u_n \in L^p(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$ and

$$F_\mu(u) \geq \limsup_{n\to\infty} F_{\mu_n}(u_n),$$

and for every convergent sequence $u_n \to u$ in $L^p(\Omega)$

$$F_\mu(u) \leq \liminf_{n\to\infty} F_{\mu_n}(u_n).$$

In Definition 2.1, by the identification of a quasi-open set A with the measure $\infty_{\Omega\setminus A}$, we implicitly have the definition of the γ-convergence of a sequence of quasi-open sets. In general, the γ-limit of a sequence of quasi-open sets is a measure of $\mathcal{M}_0^p(\Omega)$. In particular, this measure can be itself of the form $\infty_{\Omega\setminus A}$.

Note that the γ-convergence is metrizable by the following distance

$$d_p(\mu_1, \mu_2) = \int_\Omega |w_{\mu_1} - w_{\mu_2}| dx,$$

where w_μ is the variational solution of

$$-\Delta_p w_\mu + \mu |w_\mu|^{p-2} w_\mu = 1 \tag{3}$$

in $W_0^{1,p}(\Omega) \cap L^p(\Omega, \mu)$ (see [5, 15]). The precise sense of the equation is the following: $w_\mu \in W_0^{1,p}(\Omega) \cap L^p(\Omega, \mu)$ and for every $\phi \in W_0^{1,p}(\Omega) \cap L^p(\Omega, \mu)$

$$\int_\Omega |\nabla w_\mu|^{p-2} \nabla w_\mu \nabla \phi dx + \int_\Omega |w_\mu|^{p-2} w_\mu \phi d\mu = \int_\Omega \phi dx.$$

In view of the result of Hedberg [16], if A is an open subset of Ω, the solution of this equation associated to the measure $\infty_{\Omega\setminus A}$ is nothing else but the solution in the sense of distributions of

$$-\Delta_p w = 1 \text{ in } A, \quad w \in W_0^{1,p}(A).$$

Throughout the paper, by w_μ we denote the solution of (3) associated to the measure μ, and by w_A the solution of the same equation associated to the measure $\infty_{\Omega\setminus A}$.

We refer to [14] for the following result.

Proposition 2.2 *The space $\mathcal{M}_0^p(\Omega)$, endowed with the distance d_p, is a compact metric space. Moreover, the class of measures of the form $\infty_{\Omega\setminus A}$, with A open (and smooth) subset of Ω, is dense in $\mathcal{M}_0^p(\Omega)$.*

Given a measure $\mu \in \mathcal{M}_0^p(\Omega)$, we call the regular set of the measure μ the quasi-open set $\{w_\mu > 0\}$. We also notice that this set, which is denoted A_μ, coincides up to a set of zero capacity with the union of all finely open sets of finite μ measure.

Lemma 2.3 *Assume $(A_n), (B_n)$ are two sequences of quasi-open sets which γ-converge to μ_A, μ_B, respectively. If $\mathrm{cap}_p(A_n \cap B_n) = 0$ for every $n \in \mathbb{N}$, then $\mathrm{cap}(A_{\mu_A} \cap A_{\mu_B}) = 0$.*

Proof We notice that $w_{A_n} \cdot w_{B_n} = 0$ q.e. Passing to the limit a.e. and using the quasi-continuity of w_{μ_A} and w_{μ_B}, we get $w_{\mu_A} \cdot w_{\mu_B} = 0$ q.e., hence the conclusion. $\square$

On $\mathcal{M}_0^p(\Omega)$, the following monotonicity is considered on the family of measures:

$$\mu_1 \le \mu_2 \text{ if } \forall A \subseteq \Omega,\ A \,\text{quasi open}\ \mu_1(A) \le \mu_2(A).$$

We notice (see [5], [13]) that every monotone (increasing or decreasing) sequence of measures is γ-convergent.

The obstacle problem. Although the obstacle problem can be defined properly in the frame of measurable or quasi lower semicontinuous functions, in the most part of the paper we restrict ourselves to obstacles which are elements of $W_0^{1,p}(\Omega)$. Roughly speaking, for $q = p/(p-1)$, given a function $f \in W^{-1,q}(\Omega)$ and a measurable function v, the solution of the obstacle problem associated to v and f is the unique solution of the minimization problem

$$\min\{\frac{1}{p}\int_\Omega |\nabla \phi|^p - < f, \phi >_{W^{-1,q}(\Omega)\times W_0^{1,p}(\Omega)} :\ \phi \in K_v\},$$

where

$$K_v = \{\phi \in W_0^{1,p}(\Omega) : \phi \le v \text{ a.e. } \Omega\}.$$

If the obstacle v is an element of $W_0^{1,p}(\Omega)$, the inequality $\phi \le v$ in the previous set can be equivalently taken in the sense a.e. or p-q.e. (for quasi-continuous representatives). In the sequel we concentrate our attention only on obstacles belonging to $W_0^{1,p}(\Omega)$.

Definition 2.4 *Let $v_n, v \in W_0^{1,p}(\Omega)$. We say that v_n converges in the sense of obstacles to v if*

$$W_0^{1,p}(\Omega) \ni u \mapsto \int_\Omega |\nabla u|^p dx + \infty_{\{v_n\}}(u) \tag{4}$$

Γ-converges in $L^p(\Omega)$ to

$$W_0^{1,p}(\Omega) \ni u \mapsto \int_\Omega |\nabla u|^p dx + \infty_{\{v\}}(u) \tag{5}$$

where $\infty_{\{v\}}(u) = 0$ if $u \le v$ p-q.e. and $+\infty$ if not. We write $v_n \overset{obst}{\longrightarrow} v$.

The main consequence of the convergence of obstacles $v_n \overset{\text{obst}}{\longrightarrow} v$ is that for every $f \in W^{-1,q}(\Omega)$ the sequence of solutions u_n of the obstacle problem associated to v_n and f converges strongly in $W_0^{1,p}(\Omega)$ to the solution associated to v.

The convergence in the sense of obstacles is equivalent to the convergence in the sense of Mosco of K_{v_n} to K_v (see [1]), i.e. to the following two relations:

1. $\forall u \in K_v$, $\exists u_n \in K_{v_n}$ such that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$;
2. If $u_{n_k} \in K_{v_{n_k}}$ and $u_{n_k} \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, then $u \in K_v$.

Notice that if v_n converges weakly in $W_0^{1,p}(\Omega)$ to v, then the second Mosco condition is satisfied.

We recall from [6, Corollary 4.9] and [12] the following characterization of the obstacle convergence.

Theorem 2.5 *Let $v_n, v \in W_0^{1,p}(\Omega)$, $v_n \geq 0$. Then v_n converges in the sense of obstacles to v if and only if there exists a dense set $T \subseteq \mathbb{R}$ such that*

$$\{v_n > t\} \xrightarrow{\gamma} \{v > t\} \quad \forall t \in T.$$

Operators similar to the p-Laplacian. Assume that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Charatheodory function which is homogeneous of degree $p-1$ in the second variable

$$a(x, t\zeta) = |t|^{p-2} t a(x, \zeta), \quad t \in \mathbb{R}, t \neq 0. \tag{6}$$

We notice that in order to have a precise description of the γ-limits of sequences of quasi-open sets, the homogeneity property is crucial (see [9] and [15]).

We suppose as usual the monotonicity assumptions on $a(x,\zeta)$ (see for instance [15]): there exist two constants c_0, c_1 with $0 < c_0 \leq c_1 < \infty$ such that, for a.e. $x \in \Omega$ and for every $\zeta_1, \zeta_2 \in \mathbb{R}^N$ we have in the case $2 \leq p < +\infty$

$$(a(x,\zeta_1) - a(x,\zeta_2), \zeta_1 - \zeta_2) \geq c_0 |\zeta_1 - \zeta_2|^p \tag{7}$$

$$|a(x,\zeta_1) - a(x,\zeta_2)| \leq c_1 (|\zeta_1| + |\zeta_2|)^{p-2} |\zeta_1 - \zeta_2|, \tag{8}$$

and in the case $1 < p \leq 2$

$$(a(x,\zeta_1) - a(x,\zeta_2), \zeta_1 - \zeta_2) \geq c_0 (|\zeta_1| + |\zeta_2|)^{p-2} |\zeta_1 - \zeta_2|^2 \tag{9}$$

$$|a(x,\zeta_1) - a(x,\zeta_2)| \leq c_1 |\zeta_1 - \zeta_2|^{p-1}. \tag{10}$$

By the assumptions made on $a(x,\zeta)$ the operator $Au = -\text{div}\,\Big(a(x, \nabla u)\Big)$ turns out to be continuous and strongly monotone from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,q}(\Omega)$ via the pairing:

$$< Au, v >= \int_\Omega a(x, \nabla u) \cdot \nabla v dx \qquad \forall u, v \in W_0^{1,p}(\Omega). \tag{11}$$

If $a(x,\zeta) = |\zeta|^{p-2}\zeta$, then A is the p-Laplace operator

$$-\Delta_p u = -\text{div}\,(|\nabla u|^{p-2} \nabla u).$$

3 Characterization of the lower-semicontinuty

The main result of the paper is the following.

Theorem 3.1 *Let $(v_n) \subseteq W_0^{1,p}(\Omega)$, $v_n \geq 0$, $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$. The following assertions are equivalent:*

i) v_n *converges in the sense of obstacles to* v*;*

ii) *for every sequence* $(u_n) \subseteq W_0^{1,p}(\Omega)$*, such that* $-\Delta_p u_n \geq 0$ *and* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,p}(\Omega)$ *we have*

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla v_n dx \geq \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx. \tag{12}$$

Proof Necessity: *i)* $\Longrightarrow$ *ii)*. Assume *i)* holds true. Consequently K_{v_n} converges in the sense of Mosco to K_v. In particular, the first condition of the Mosco convergence, insures the existence of a sequence $\theta_n \in W_0^{1,p}(\Omega)$ such that $\theta_n \leq v_n$ and $\theta_n \longrightarrow v$ in $W_0^{1,p}(\Omega)$-strong.

Thus, we have

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla v_n dx = \liminf_{n\to\infty} < -\Delta_p u_n, v_n >_{W^{-1,q}(\Omega)\times W_0^{1,p}(\Omega)}$$

$$\geq \liminf_{n\to\infty} < -\Delta_p u_n, \theta_n >_{W^{-1,q}(\Omega)\times W_0^{1,p}(\Omega)}$$

$$= \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \theta_n dx = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx.$$

For the last equality, we notice, on the one hand, that $|\nabla u_n|^{p-2}\nabla u_n$ converges weakly to $|\nabla u|^{p-2}\nabla u$ in $L^q(\Omega, \mathbb{R}^N)$ since $-\Delta_p u_n$ are positive and uniformly bounded Radon measures. This result is due to Boccardo and Murat [2] (see also [15, Theorem 2.10]) and is related to the pointwise convergence of the gradients. On the second hand, we have that θ_n converges strongly in $W_0^{1,p}(\Omega)$.

Sufficiency: *ii)* $\Longrightarrow$ *i)*. Assume *ii)* holds. We shall prove that v_n converges in the sense of obstacles to v. We rely both on the characterization of the obstacle convergence via the Mosco convergence, and since $v_n \geq 0$, on the characterization through the γ-convergence of the level-sets $\{v_n > t\}$. (Theorem 2.5 in Section 2).

We notice that from the weak convergence in $W_0^{1,p}(\Omega)$, $v_n \rightharpoonup v$, the second Mosco condition of the desired Mosco convergence $K_{v_n} \xrightarrow{M} K_v$ is automatically satisfied. Moreover, the first Mosco condition has to be proved only for the function v. Indeed, if there exists $\theta_n \leq v_n$ such that $\theta_n \longrightarrow v$ in $W_0^{1,p}(\Omega)$-strong, then for every $\psi \leq v$, the first Mosco condition holds with the sequence $\min\{\theta_n, \psi\} \leq v_n$.

Assume for contradiction that $\exists \delta > 0$ and a subsequence (v_{n_k}) such that

$$\liminf_{k\to\infty} \min_{u\in K_{v_{n_k}}} \|u - v\|_{W_0^{1,p}(\Omega)} \geq \delta > 0. \tag{13}$$

In order to achieve the contradiction in assumption (13), we construct a subsequence of (v_{n_k}), which converges in the sense of obstacles to v.

Step 1. We construct a mapping

$$[0, +\infty) \mapsto \mu_t \in \mathcal{M}_0^p(\Omega),$$

which is increasing in the sense of measures of $\mathcal{M}_0^p(\Omega)$ and such that for a subsequence of (v_{n_k}) (denoted with the index r) and for a dense set $T \subseteq [0,+\infty)$ we have

$$\forall t \in T \;\; \{v_r > t\} \xrightarrow{\gamma} \mu_t.$$

The construction of the mapping $t \mapsto \mu_t$ is done by a diagonal procedure using the compactness and the metrizability of the γ-convergence in $\mathcal{M}_0^p(\Omega)$. We follow the approach used in [7] for constructing the relaxed space for obstacles. Let $T = \mathbb{Q} \cap \mathbb{R}^+ = \{t_1, t_2, .., t_k, ..\}$. For $r = 1, ...,$ we successively extract γ-convergent subsequences for the sequences of the quasi-open sets $\{v_{n_k} > t_r\}$, and define μ_{t_r} being the γ-limit. By a diagonal procedure, the metrizability of the γ-convergence gives the existence of a subsequence of (v_n) and of a family of measures (μ_{t_r}) such that the level sets $\{v_{n_k} > t_r\}$ γ-converges to μ_{t_r}.

Using the density of the set T in $[0,+\infty)$ and relying both on the monotonicity of the measures and the γ convergence of monotone sequences, we define the mapping $t \mapsto \mu_t$ by γ-continuity on the right. Finally, it can be concluded as in [6] that the convergence $\{v_r > t\} \xrightarrow{\gamma} \mu_t$ holds on $\mathbb{R} \setminus N$, where N is at most countable. Precisely, N is the set of discontinuity points of the monotonous real function $[0,+\infty) \ni t \to \int_\Omega w_{\mu_t} \in \mathbb{R}$.

By abuse of notation and for the simplicity of the exposition, we renote the constructed subsequence by (v_n).

We use the hypothesis *ii)* and choose in relation (12) the sequence (u_n) defined in the following way:

$$\begin{cases} -\Delta_p u_n &= 0 \;\text{ in }\; \{v_n > t\} \\ \quad u_n &= w_\Omega \;\text{ in }\; \Omega \setminus \{v_n > t\} \end{cases} \tag{14}$$

Following [17], we have $-\Delta_p u_n \geq 0$ in Ω, so the sequence (u_n) is admissible in *ii)*. This sequence is bounded in $W_0^{1,p}(\Omega)$, hence for a subsequence, still denoted using the same index, we have that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. Following [2], we also have that $\nabla u_n \to \nabla u$ a.e. in Ω, since $-\Delta_p u_n$ are uniformly bounded positive Radon measures.

The information on the γ-convergence of the level sets $\{v_n > t\}$ gives

- $u = w_\Omega$ on $\Omega \setminus A_\mu$. This is a consequence of the fact that $u - w_\Omega \in L^p(\Omega, \mu)$.
- On A_μ, u satisfies the equation

$$-\Delta_p u + \mu |u - w_\Omega|^{p-2}(u - w_\Omega) = 0 \tag{15}$$

in the variational sense of $W_0^{1,p}(\Omega) \cap L^p(\Omega, \mu)$, i.e. for every $\phi \in W_0^{1,p}(\Omega) \cap L^p(\Omega, \mu)$ we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi dx + \int_\Omega |u - w_\Omega|^{p-2}(u - w_\Omega)\phi d\mu = 0. \tag{16}$$

Indeed, for proving that u satisfies equation (15) on A_μ with the measure μ issued from the γ-convergence of the level sets $\{v_n > t\}$, we can use a similar argument as in [9]. Denoting $\theta_n = u_n - w_\Omega$ and $\theta = u - w_\Omega$, it is enough the prove that

$$g_n =: \Delta_p(\theta_n + w_\Omega) - \Delta_p \theta_n \xrightarrow{\mathcal{H}} g =: \Delta_p(\theta + w_\Omega) - \Delta_p \theta,$$

the convergence $\mathcal{H}$ being understood in the following sense: for every sequence $\phi_n \in W_0^{1,p}(\{v_n > t\}$ which converges weakly in $W_0^{1,p}(\Omega)$ to ϕ we have

$$< g_n, \phi_n >_{W^{-1,q}(\Omega)\times W_0^{1,p}(\Omega)} \rightarrow < g, \phi >_{W^{-1,q}(\Omega)\times W_0^{1,p}(\Omega)} .$$

Since we know that ∇u_n converges a.e. to ∇u, for every $\delta > 0$, there exists a set E such that $|E| < \delta$ and

$$\nabla u_n \to \nabla u$$

uniformly on $\Omega \setminus E$. The same holds for $\nabla \theta_n$ and $\nabla \theta$. Consequently

$$\lim_{n\to\infty} \Big| \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla \theta_n|^{p-2}\nabla\theta_n)\nabla\phi_n dx - \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta)\nabla\phi dx \Big|$$

$$\leq \limsup_{n\to\infty} \int_E |(|\nabla u_n|^{p-2}\nabla u_n - |\nabla\theta_n|^{p-2}\nabla\theta_n)\nabla\phi_n| dx + \int_E |(|\nabla u|^{p-2}\nabla u - |\nabla\theta|^{p-2}\nabla\theta)\nabla\phi| dx.$$

For every $\varepsilon > 0$, there exists M such that

$$|\nabla\rho| \geq M \implies \Big| |\nabla(\rho + w_\Omega)|^{p-2}\nabla(\rho + w_\Omega) - |\nabla\rho|^{p-2}\nabla\rho \Big| \leq \varepsilon |\nabla\rho|^{p-1}.$$

Thus, for a given $\varepsilon > 0$ we have

$$\int_E |(|\nabla u_n|^{p-2}\nabla u_n - |\nabla\theta_n|^{p-2}\nabla\theta_n)\nabla\phi_n| dx$$

$$\leq \varepsilon \int_{E\cap\{|\nabla\theta_n|\geq M\}} |\nabla\theta_n|^{p-1}|\nabla\phi_n| dx + 3M^{p-1}\int_{E\cap\{|\nabla\theta_n|<M\}} |\nabla\phi_n| dx$$

$$\leq C(\varepsilon + 3M^{p-1}|E \cap \{|\nabla\theta_n| < M\}|^{\frac{1}{q}}).$$

Taking the limit $n \to \infty$, making successively $\delta \to 0$ and $\varepsilon \to 0$, we conclude that $g_n \xrightarrow{\mathcal{H}} g$.

In order to prove that u satisfies equation (15) on A_μ, we use the fact that

$$-\Delta_p \theta_n = g_n \text{ in } \{v_n > t\},$$

$$\theta_n \in W_0^{1,p}(\{v_n > t\}),$$

that $\{v_n > t\}$ γ-converges to μ and $g_n \xrightarrow{\mathcal{H}} g$.

Applying inequality (12) to the sequence (u_n) constructed above, we get

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v_n dx \geq \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx,$$

or, decomposing the integrals by using $v_n - t = (v_n - t)^+ - (v_n - t)^-$,

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(v_n - t)^+ dx - \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla(v_n - t)^- dx \geq$$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(v - t)^+ dx - \int_\Omega |\nabla u|^{p-2}\nabla u\nabla(v - t)^- dx.$$

We have that $u_n = w_\Omega$ on $\{v_n > t\}$, hence we get

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla (v_n - t)^- dx \to \int_\Omega |\nabla u|^{p-2} \nabla u \nabla (v - t)^- dx.$$

Consequently

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla (v_n - t)^+ dx \geq \int_\Omega |\nabla u|^{p-2} \nabla u \nabla (v - t)^+ dx.$$

But, u_n is p-harmonic on $\{v_n > t\}$, hence the integrals on the left hand side are equal to zero! On the right hand side, we use the equation (16) satisfied by u on A_μ and get

$$0 \geq \int_\Omega |u - w_\Omega|^{p-2} (w_\Omega - u)(v - t)^+ d\mu. \tag{17}$$

Since all terms under the sum are positive on the right hand side, we get

$$\int_\Omega |u - w_\Omega|^{p-2} (w_\Omega - u)(v - t)^+ d\mu = 0.$$

By the comparison principle of p-superharmonic functions we know $w_\Omega(x) > u(x)$ p-q.e. on A_μ. Consequently we get that $\mu(\{v > t\}) = 0$.

The main idea is to prove that $\mu(A_\mu) = 0$ in which case $\mu = \infty_{\Omega \setminus A_\mu}$ and thus $A_\mu = \{v > t\}$. As a consequence we would get that the sequence of level sets $\{v_n > t\}$ γ-converges to $\{v > t\}$ and Theorem 2.5 could be applied. It may be possible that $\{v > t\}$ is a strict subset of A_μ (in the sense of capacity), so this argument is not enough to conclude that $\mu = 0$ on A_μ, and needs further investigation.

Step 2. From the weak-$W_0^{1,p}(\Omega)$ convergence $v_n \rightharpoonup v$, we get

$$(v_n - t)^+ \rightharpoonup (v - t)^+$$

weakly in $W_0^{1,p}(\Omega)$, and from the γ-convergence of the sets $\{v_n > t\}$ to μ we get $(v - t)^+ \in W_0^{1,p}(\Omega) \cap L^p(\Omega, \mu)$. The same holds also for $(v - (t + \varepsilon))^+$ for $\varepsilon > 0$. Let us denote

$$A_t = \gamma - \lim_{\varepsilon \to 0} \{v > t + \varepsilon\}.$$

This γ-limit exists from the monotonicity of the sets. Obviously we get $A_t \subseteq A_\mu$.

On the other hand, following Lemma 2.3, $A_\mu \cap \{v < t\} = \emptyset$ in the sense of capacity, hence $A_\mu \subseteq \{v \geq t\}$. If t is a γ-continuity point for the mapping $t \mapsto \{v > t\}$ we get that $A_\mu = \{v > t\}$ up to a set of zero capacity.

Step 3. We conclude the proof by noticing that the mapping $t \mapsto \{v > t\}$ is γ-continuous on $\mathbb{R}$ with the exception of an at most countable family of points, hence we get that $\mu_t = \infty_{\{v \leq t\}}$ on $\mathbb{R}$ with the exception of an at most countable family of points, so the limit in the sense of obstacles of the sequence (v_n) is the function v.

□

Remark 3.2 In [3] the authors prove that if u_n, v_n are weakly convergent sequences of non negative functions of $W_0^{1,p}(\Omega)$ such that $-\Delta_p u_n \geq 0$ and $-\Delta_p v_n \geq 0$, then relation (1) holds true. The main technical argument is that $\min\{v_n, v\}$ converges strongly in $W_0^{1,p}(\Omega)$ to v, which is obtained as a consequence of the p-superharmonicity of v_n. We notice that the strong convergence $\min\{v_n, v\} \to v$ together with the weak convergence $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ imply that $v_n \xrightarrow{obst} v$, hence assertion i) of Theorem 3.1 is satisfied.

4 Futher remarks

The extension of the results of the paper to higher order operators is not an obvious matter. On the one hand, the positivity preserving property for higher order operators is depending on the geometric set where the operator is defined. For the bi-Laplace operator, positivity preserving holds, for example, on balls and the entire space but fails in general, even on smooth sets (see [11]). If the operators are positivity preserving, the sufficiency part of Theorem 3.1 still holds true. Nevertheless, dealing with the necessity part is more difficult, since the convergence of obstacles and the relaxation of sets through γ-convergence is not known. Moreover, the lack of reticularity of the Sobolev spaces of order greater than 1 may be a supplementary difficulty for the necessity part.

Remark 4.1 There is a significant non-symmetry between the two terms in the duality product. The convergence in the sense of the obstacles of v_n is related to the Mosco convergence of the sets K_{v_n} and is independent on the choice of the operator itself which is associated to the terms u_n. This means that if Theorem 3.1 holds for a sequence $(v_n)_n$ and the p-Laplace operator, then Theorem 3.1 holds for the same sequence $(v_n)_n$ and an operator similar to the p-Laplacian.

In the sufficient condition given in [3], the superharmonicity of the first sequence u_n serves for monotonicity of the duality product of $-\Delta_p u_n$ against v_n and the metric projection of v on the cone $\{\varphi \leq v_n\}$, respectively, and on the other hand, the p-superharmonicity of the second sequence v_n is a sufficient condition to obtain the obstacle convergence as a consequence of the weak convergence.

Remark 4.2 (Uniformly oscillating obstacles) Assume that $N \geq p > N-1$ and $v_n \in W_0^{1,p}(\Omega)$, $v_n \geq 0$ are continuous and have uniform oscillations, i.e. there exists a sequence of numbers $(l_n)_n$ and a dense set $\{t_1, t_2, ...\} \subseteq \mathbb{R}_+$ such that

$$\forall r, n \in \mathbb{N},\ \sharp\{v_n \leq t_r\} \leq l_r, \tag{18}$$

where $\sharp A$ denotes the number of the connected components of the set A. If $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$, then v_n converges in the sense of obstacles to v. Indeed, the second Mosco condition is satisfied as a consequence of the weak convergence $v_n \rightharpoonup v$. In order to prove the first Mosco condition, we follow the idea of the proof of Theorem 3.1, and assume for contradiction that relation (13) holds for some $\delta > 0$. From (18) and the shape compactness/stability result of [8], there exists a subsequence of the sets $\{v_n > t_r\}$ and a set Ω_{t_r}, such that $\{v_{n_k} > t_r\}$ γ-converges to Ω_{t_r}. Consequently, by a diagonal extraction procedure as in Theorem 3.1, we construct an obstacle h such that $\{h > t_r\} = \Omega_{t_r}$ and the sequence v_{n_k} converges in the sense of obstacles to h. From the γ-convergence of the level sets, we get $(v - t_r)^+ \in W_0^{1,p}(\Omega_{t_r})$, hence $h \geq v$. This contradicts (13), from the Mosco convergence.

It is needless to say that v_n are not, in general, p-superharmonic.

Remark 4.3 In [3], the authors give an example of a sequence of positive functions v_n which are not superharmonic such that inequality (1) fails to be true for a suitable sequence u_n of superharmonic functions. This construction is done around the pioneering result of Cioranescu and Murat [10] on the "strange term" appearing in the relaxation process through the γ-convergence. The presence of the strange term is an argument of non γ-convergence of obstacles! So, from this point of view it is not surprising that inequality (1) is violated, although the choice of u_n has to be done carefully.

Remark 4.4 We give in the sequel an example of non superharmonic functions v_n for which the second assertion of Theorem 3.1 holds. Let $\eta \in C^1(\mathbb{R}^2, \mathbb{R})$ be a periodic function of period (l_1, l_2) and $\varphi \in C_0^\infty(\Omega)$. We consider the sequence of functions

$$v_n = w_\Omega + \varphi \varepsilon \eta(\frac{x}{\varepsilon}).$$

It is clear that this sequence converges weakly but not strongly in $H_0^1(\Omega)$ to $v = w_\Omega$, provided that φ or η are not the zero functions and $\varepsilon \downarrow 0$. Inequality (1) is satisfied for all admissible sequences (u_n). This is a consequence of the obstacle convergence of v_n towards v which can be easily proved. Nevertheless for small ε, the functions v_n are not superharmonic!

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