The Saint-Venant inequality for the Laplace operator with Robin boundary conditions*

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Abstract

This survey paper is focused on the Saint-Venant inequality for the Laplace operator with Robin boundary conditions. In a larger context, we make the point on the recent advances concerning isoperimetric inequalities of Faber-Krahn type for elastically supported membranes and describe the main ideas of their proofs in both contexts of rearrangement and free discontinuity techniques.

Keywords: Faber-Krahn inequality, Saint-Venant inequality, shape optimization, Robin-Laplacian

1 Introduction

The isoperimetric property of the ball concerning the first eigenvalue of the Dirichlet-Laplacian, conjectured for plane domains by Lord Rayleigh in 1877, and proved independently by Faber and Krahn in the 1920's, states that if $\Omega \subseteq \mathbb{R}^N$ is open and bounded, then

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega), \tag{1}$$

where Ω^* is a ball such that $|\Omega^*| = |\Omega|$. Here $\lambda_1(\Omega)$ is defined as the lowest value for which the problem

$$\begin{cases} -\Delta u = \lambda_1(\Omega) u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a non trivial solution. Following the review paper [21] (see also [3]), Lord Rayleigh was motivated in his conjecture by the study of the principal frequency of vibration of a plane elastic membrane fixed at its boundary, stating that the circular shape has the lowest mode of vibration (and giving some evidence of it).

Inequality (1) is usually referred to as the Faber-Krahn inequality for the first eigenvalue of the Dirichlet-Laplacian. When Ω has irregular boundary, the eigenvalue problem should be interpreted in the weak sense of Sobolev functions $W_0^{1,2}(\Omega)$ vanishing at the boundary.

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The modern approach to the proof of the Faber-Krahn inequality is due to PÓLYA and SZEGÖ and it is described in their book [22]. It relies on the *spherically symmetric decreasing rearrangement* technique applied to the expression of $\lambda_1(\Omega)$ as the Rayleigh quotient

$$\lambda_1(\Omega) = \min_{u \in W_0^{1,2}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx}.$$

Considering the first eigenfunction $u \in W_0^{1,2}(\Omega)$, one obtains a radial symmetric decreasing function $u^* \in W_0^{1,2}(B)$ equimeasurable with u (so that L^p-norms are preserved) such that

$$\int_{\Omega^*} |\nabla u^*|^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx,$$

so that inequality (1) readily follows since $\lambda_1(\Omega^*)$ is lower than the Rayleigh quotient of u^*. The properties of the spherically symmetric decreasing rearrangement show moreover that equality holds in (1) if and only if Ω is equivalent to a ball up to negligible sets.

Such an approach provides easily the validity of a whole family of Faber-Krahn inequalities: setting for $1 \leq q < \frac{2N}{N-2}$

$$\lambda_{1,q}(\Omega) := \min_{u \in W_0^{1,2}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2 \, dx}{\left(\int_\Omega |u|^q \, dx\right)^{\frac{2}{q}}}, \tag{2}$$

then again

$$\lambda_{1,q}(\Omega^*) \leq \lambda_{1,q}(\Omega) \tag{3}$$

and equality holds if and only if Ω is a ball up to negligible sets.

For plane (simply connected) domains, the case $q = 1$ is relevant in the elasticity theory of beams, and goes under the name of *torsion rigidity problem* (see e.g. [23, Section 35]): the inverse of $\lambda_{1,1}(\Omega)$ is proportional to the *torsional rigidity* of a beam with cross section Ω (here u has the meaning of a *stress function*, its derivatives being connected with the elastic forces inside the beam). That the shape of the cross section which provides the greatest torsional rigidity (under an area constraint) should be a circle was conjectured by SAINT-VENANT[1] in 1856.

In any dimension of the space, if $q = 1$, a suitably chosen minimizer u_Ω of (2) solves the torsion problem for the Dirichlet-Laplacian

$$\begin{cases} -\Delta u_\Omega = 1 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \partial\Omega. \end{cases}$$

The torsional rigidity of Ω is then $P(\Omega) = \int_\Omega u_\Omega dx$ and equals $\frac{1}{\lambda_{1,1}(\Omega)}$. It is important to notice, that the function u_Ω is also the unique minimizer of the torsional energy

$$E(\Omega) = \min_{v \in W_0^{1,2}(\Omega)} \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \int_\Omega v dx,$$

and that $P(\Omega) = -2E(\Omega)$.

The Sant-Venant inequality reads

$$P(\Omega) \leq P(\Omega^*),$$

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while equality holds if and only if Ω is a ball, up to a set of zero capacity. Equivalently, the Saint-Venant inequality can be written

$$E(\Omega^*) \le E(\Omega).$$

Along with the spherical rearrangement proof of (3) which works for every value of $q \in [1, \frac{2N}{N-2})$, it is interesting to recall a second proof argument, which is due Talenti [24] and is suitable for the case $q = 1$. He proved the following pointwise inequality

$$u_\Omega^*(x) \le u_{\Omega^*}(x), \text{a.e. } x \in \Omega^*.$$

Consequently, the Saint-Venant inequality holds by integration, since

$$\int_\Omega u_\Omega dx = \int_{\Omega^*} u_\Omega^* dx \le \int_{\Omega^*} u_{\Omega^*} dx.$$

Robin boundary conditions. This paper is focused on the Saint-Venant inequality for the Laplace operator with Robin boundary conditions, in the context of more general Faber-Krahn inequalities.

Let $\beta > 0$ and $\Omega \subseteq \mathbb{R}^N$ be an open, bounded, Lipschitz set. We consider the torsion problem for an elastically supported membrane

$$\begin{cases} -\Delta u_\Omega = 1 & \text{in } \Omega, \\ \frac{\partial u_\Omega}{\partial n} + \beta u_\Omega = 0 & \text{on } \partial\Omega \end{cases}$$

which has as unique solution u_Ω, the minimizer of

$$E_\beta(\Omega) := \min_{u \in W^{1,2}(\Omega)} \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\beta}{2}\int_{\partial\Omega} u^2 dx - \int_\Omega u dx. \tag{4}$$

We denote $P_\beta(\Omega) = \int_\Omega u_\Omega dx = -2E_\beta(\Omega)$. The main result is the following Saint-Venant inequality.

Theorem 1.1 *For every $\beta > 0$, for every bounded Lipschitz set $\Omega \subseteq \mathbb{R}^N$, the following inequality holds*

$$P_\beta(\Omega) \le P_\beta(\Omega^*),$$

with equality if and only if Ω is the ball.

In terms of torsion energies, this inequality is written: $E_\beta(\Omega^*) \le E_\beta(\Omega)$, or equivalently

$$\forall u \in W^{1,2}(\Omega) : \qquad E_\beta(\Omega^*) \le \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\beta}{2}\int_{\partial\Omega} u^2 d\mathcal{H}^{N-1} - \int_\Omega u dx.$$

There are several important points to be noticed. The inequality above can not be proved by a spherical rearrangement argument as for Dirichlet boundary conditions. This is due to the fact that u_Ω is, in general, not constant on $\partial\Omega$. Consequently, the spherical rearrangement does not control the gradient part of the energy. As well, an argument similar to the one of Talenti has not been proved to hold in this case (but nor its failure, see the last section of the paper).

The only rearrangement proof known to work for Robin boundary conditions is due to Bossel [6] and Daners [16] and works only in the case of the Faber-Krahn inequality for the first Robin eigenvalue of the Laplacian (see Section 2 for details). It appears that the technique of Bossel-Daners fails for the Saint-Venant inequality.

In order to be well posed, the torsion problem for the Laplace operator with Robin boundary conditions requires some regularity of the domain Ω. Indeed, even in the energetic formulation given in (4) one needs, at a first sight, to be able to understand the trace of a test function on $\partial\Omega$, to have a Poincaré inequality in $W^{1,2}(\Omega)$ and the compact embedding of $W^{1,2}(\Omega)$ in $L^1(\Omega)$. This is the reason for which in Theorem 1.1 the hypothesis on the Lipschitz regularity of Ω is required. In fact, this restriction (which a priori is reasonable), is not necessary at all. Understanding how to remove this hypothesis is the key point of the proof of Theorem 1.1 by variational techniques, and it will lead to a strengthening of the optimality of the ball. Indeed, the Saint-Venant inequality holds in a much more general setting, for arbitrary domains. In Section 3 we detail this point.

For now, let us give an intuitive approach to arbitrary domains. Let $\Omega \subseteq \mathbb{R}^N$ be an arbitrary open set of finite measure. Even if the trace of a Sobolev function $u \in W^{1,2}(\Omega)$ is not defined, if one restricts to functions $u \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$, the quantity

$$\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\beta}{2}\int_{\partial\Omega} u^2 d\mathcal{H}^{N-1} - \int_\Omega u dx$$

is well defined (at least equal to $+\infty$). As a consequence, the question

$$\forall u \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega}) : \qquad E_\beta(\Omega^*) \le \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\beta}{2}\int_{\partial\Omega} u^2 d\mathcal{H}^{N-1} - \int_\Omega u dx \ ?$$

is well posed. This approach amounts to the definition of the Robin problem in arbitrary domains introduced by Daners [15] which is based on the Maz'ja space $W^{1,2}(\Omega, \partial\Omega)$. In Section 3 we analyse this issue and show how this setting is covered by our method.

2 The Robin-Laplacian: Faber-Krahn and Saint-Venant inequalities

Let $\beta > 0$ be fixed and $\Omega \subseteq \mathbb{R}^N$ be an open, bounded, Lipschitz, connected set. For every $q \in [1, 2]$ we define

$$\lambda_{1,q}^\beta(\Omega) = \min_{\substack{u \in W^{1,2}(\Omega)\\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} u^2 d\mathcal{H}^{N-1}}{\displaystyle\Big(\int_\Omega |u|^q\Big)^{\frac{2}{q}} dx}. \tag{5}$$

For $q = 1$, we find the torsional rigidity defined in (4), with $\lambda_{1,1}^\beta(\Omega) = \frac{1}{P_\beta(\Omega)}$.

For $q = 2$ we get the first eigenvalue of the Robin-Laplacian, in which case the minimizer $u \in W^{1,2}(\Omega)$ solves in a weak sense the equation

$$\begin{cases} -\Delta u = \lambda_{1,2}^\beta(\Omega) u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

The function u is of constant sign, since Ω is connected.

Here is stated a general result proved in [10].

Theorem 2.1 (A family of Faber-Krahn inequalities) *For every $\beta > 0$, for every $q \in [1, 2]$, for every bounded open Lipschitz set $\Omega \subseteq \mathbb{R}^N$*

$$\lambda_{1,q}^\beta(\Omega) \ge \lambda_{1,q}^\beta(\Omega^*),$$

with equality if and only if Ω is a ball.

In particular, for $q = 1$ this is the Saint-Venant inequality which provides the result for Theorem 1.1.

For the case $q = 2$, Theorem 2.1 has been proved by Bossel in 1986 in two dimensions of the space for simply connected smooth open sets [6]. The idea relies on a derangement procedure and the study of a functional (called in the sequel H-functional) defined on the level sets of an eigenfunction. This idea was completed and extended by Daners [16] in any dimension of the space and to Lipschitz sets. Up to the knowledge of the authors, no suitable modification of this method appears to be successful in the case when $q \neq 2$. On the contrary, if instead of the Laplace operator one considers the p-Laplace operator, the H-functional can be easily adapted (see [7, 14]) but again this requires the same p-norm for both numerator and denominator.

In [4], for $q = 1$, Bandle and Wagner proved that the ball is a local maximizer of the torsional rigidity, for deformations by smooth vector fields. This proof is based on the fact that the first order shape derivative of the torsional rigidity is vanishing for vector fields preserving the measure, while the second order shape derivative is strictly negative at the ball.

The complete proof of Theorem 2.1 was given in [10] in a larger setting than just the Lipschitz sets. The method is based on a free discontinuity approach to the isoperimetric inequality via the principle "*existence and regularity $\Longrightarrow$ the optimum is the ball*" (a free discontinuity approach for the linear eigenvalue is already contained in [9], but the optimality of the ball was shown by adapting the Bossel-Daners method).

The purpose of this survey article is to give the main ideas, from an intuitive perspective, of the proof of Bossel-Daners for $q = 2$ and of the proof given in [10] for $q = 1$. In fact, the restriction to $q = 1$ simplifies formally only the exposition of the proof of Theorem 2.1. Nevertheless, without any significant difference the same arguments hold for every $q \in [1, 2]$.

2.1 The proof of Theorem 2.1 for $q = 2$ by the method of Bossel-Daners.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded connected open set of class C^2. Let u be a nonzero eigenfunction associated to the first eigenvalue $\lambda_{1,2}^{\beta}(\Omega)$, which is smooth. We can assume that $u \geq 0$ in Ω. In fact, one can easily notice that $u > 0$ on $\overline{\Omega}$ as a consequence of the Hopf principle. Indeed, since u is superharmonic in Ω, at any point $x_0 \in \partial\Omega$ where u attains its minimum one gets that the normal derivative can not vanish: hence $u(x_0) > 0$ so that $\inf_{\overline{\Omega}} u > 0$. Moreover, it can be easily noticed that u is bounded.

For every $t > 0$ we denote $U_t := \{u > t\}$. Let $\Phi : \Omega \to \mathbb{R}^+$ be a bounded measurable function. We introduce, the H-functional defined for every couple (U_t, Φ) by

$$H(U_t, \Phi) = \frac{1}{|U_t|}\Big(\int_{\partial_i U_t} \Phi d\mathcal{H}^{N-1} + \int_{\partial_e U_t} \beta d\mathcal{H}^{N-1} - \int_{U_t} |\Phi|^2 dx \Big),$$

where $\partial_i U_t = \partial U_t \cap \Omega$ and $\partial_e U_t = \partial U_t \cap \partial\Omega$.

The main properties of the H-functional are the following (see [16]):

(A) $\forall t \in (0, \|u\|_\infty) \, : \, H(U_t, \frac{|\nabla u|}{u}) = \lambda_{1,2}^{\beta}(\Omega)$;

(B) $\forall \Phi, \exists t \in (0, \|u\|_\infty) \, : \, \lambda_{1,2}^{\beta}(\Omega) \geq H(U_t, \Phi)$

Property (A) is obtained by taking $\frac{1}{u}$ as a test function in the equation (6) and summing over U_t. Property (B) is obtained by contradiction: assuming the converse inequality for every t, one gets that the function

$$t \mapsto t^2 \int_{U_t} \Big(\Phi - \frac{|\nabla u|}{u}\Big) \frac{|\nabla u|}{u} dx$$

is non increasing on $(0,\|u\|_\infty)$. Having zero limits at both 0 and $\|u\|_\infty$, one gets a contradiction.

Let us denote by B the ball of the same measure as Ω and by u_B a positive eigenfunction on the ball. It is very easy to notice that u_B is radially symmetric and that $\frac{|\nabla u_B|}{u_B} \leq \beta$ in B. We dearrange the function $\frac{|\nabla u_B|}{u_B}$ on the level sets $(U_t)_t$, and denote this function ${}^*\Phi$. One picks the value t given by property (B) for the function ${}^*\Phi$ and writes the following chain of inequalities

$$\lambda_{1,2}^{\beta}(\Omega) \geq H(U_t, {}^*\Phi) \geq H\left(B_{t*}, \frac{|\nabla u_B|}{u_B}\right) = \lambda_{1,2}^{\beta}(B),$$

where t^* is chosen such that the measure of U_t equals to the measure of $\{u_B > t^*\}$. The last inequality is a direct consequence of the inequality $\frac{|\nabla u_B|}{u_B} \leq \beta$ and the properties of the rearrangement: the L^2 norm of the function is preserved while the perimeter of U_t is larger than the perimeter of the ball $\{u_B > t^*\}$. The last equality, is a consequence of property (A) on the ball.

The passage from a smooth set Ω to a Lipschitz set is done by approximation: every bounded Lipschitz set is the Hausdorff limit of a sequence of smooth C^2 domains, such that the normals converge in a suitable sense (locally in systems of coordinates) (see [16]).

Remark 2.2 The proof of Bossel-Daners based on the properties of the H-functional above, can be easily extended with no major modification to Lipschitz sets and to the p-Laplacian, i.e. for minimizers of

$$\min_{\substack{u\in W^{1,p}(\Omega)\\ u\neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega |u|^p dx}.$$

The passage to the p-Laplacian requires the analysis of the eigenfunction on the ball: namely that the solution is radial and that the maximal ratio of $\frac{|\nabla u_B|}{u_B}$ is attained at the boundary. On the other hand, the passage to Lipschitz sets is done by handling the test function $\frac{1}{u}$ with more care, since it lacks of smoothness. We refer the reader to [14, 7] for more details.

Remark 2.3 Up to the knowledge of the authors, the proof based on the H-functional can not be extended to the Saint-Venant inequality, or to any other $\lambda_{1,q}^{\beta}$, with $q \neq 2$. A suitable modification of the H-functional was not (yet) found. In fact, the nonlinear character of the Rayleigh quotient in the case $q \neq 2$, as a consequence of the different norms appearing on the fraction, makes that the properties (A), (B) of the H-functional fail to be true.

Following the proof of Bossel-Daners of Theorem 2.1 in the case $q = 2$ on Lipschitz sets, several questions remained unanswered after the publication of [16]:

- Can the Faber-Krahn inequality hold on arbitrary open sets ? As for the torsion problem, it is not clear what is the correct way to understand Robin conditions in an arbitrary open set. The validity of the Faber-Krahn inequality is a well posed question with a non-trivial answer if it is set in the following way: given a bounded open set Ω, is the following inequality true

$$\forall u \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega}), u \neq 0 \,:\, \frac{\displaystyle\int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} u^2 d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega |u|^2 dx} \geq \lambda_{1,2}^{\beta}(\Omega^*)? \qquad (7)$$

  More generally, if a pointwise trace on $\partial\Omega$ can be defined for any function $u \in W^{1,2}(\Omega)$, the validity of the inequality above is still a well posed question replacing $W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ by

$W^{1,2}(\Omega)$. If Ω is for instance a Lipschitz set from which one removes a Lipschitz crack (so that the trace is defined on both sides of the crack), or if Ω is an open set with a piecewise smooth boundary having a finite number of cusps, the question above is still well posed for any function $u \in W^{1,2}(\Omega)$. Summarizing, if locally $\mathcal{H}^{N-1}$-a.e. the trace at the boundary can be defined in some sense, the problem above is still well posed even though $W^{1,2}(\Omega)$ is not compactly embedded in $L^2(\Omega)$ and the (pointwise) trace is not continuous from $W^{1,2}(\Omega)$ into $L^2(\partial\Omega)$.

In other words, is $\lambda_{1,2}^{\beta}(\Omega^*)$ the worst constant for the Poincaré inequality with trace term

$$\forall u \in W^{1,2}(\Omega) \, : \, \int_{\Omega} |\nabla u|^2 dx + \beta \int_{\partial\Omega} u^2 d\mathcal{H}^{N-1} \geq \lambda_{1,2}^{\beta}(\Omega^*) \int_{\Omega} |u|^2 dx?$$

- For $q \neq 2$ (in particular for $q = 1$) is the following inequality true (in the family of Lipschitz sets, or in a more general setting as described above)

$$\lambda_{1,q}^{\beta}(\Omega) \geq \lambda_{1,q}^{\beta}(\Omega^*) \; ?$$

Remark 2.4 (The Robin problem on arbitrary open sets.) We recall in the sequel the perspective of Daners [15] (see also Arendt and Warma [2]) to define the Robin problem in a non smooth setting. Let Ω be an arbitrary bounded open set. The way followed in [15] to define the Robin problem is to replace the classical Sobolev space $W^{1,2}(\Omega)$ by the so called Maz'ja space $W^{1,2}(\Omega, \partial\Omega)$ which is defined as the completion of $W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ for the norm $|\cdot|_{W^{1,2}(\Omega)} + |\cdot|_{L^2(\partial\Omega, \mathcal{H}^{N-1})}$.

It turns out that $W^{1,2}(\Omega, \partial\Omega)$ is, in general, a subspace in $W^{1,2}(\Omega)$ and coincides sometimes with $W^{1,2}(\Omega)$, provided that Ω has some smoothness, as for example it is Lipschitz regular. This space has some good properties, but also some important inconvenients. Here are the good properties:

- It is a well defined Hilbert space for every bounded open set, which coincides with the "true" Sobolev space provided some smoothness of the boundary is available.
- The embedding of $W^{1,2}(\Omega, \partial\Omega)$ in $L^2(\Omega)$ is compact, whatever the regularity of Ω is.
- Every function from $W^{1,2}(\Omega, \partial\Omega)$ has a trace at $\partial\Omega$. Inequality (7) is well posed for every function $u \in W^{1,2}(\Omega, \partial\Omega)$, in view of the existence of a trace term.

The main inconvenients of the space have an impact on the interpretation of the Robin boundary conditions in a non-smooth setting and on the lack of geometric stability of the Robin problem in this setting. Here we list the most important ones.

- A function $u \in W^{1,2}(\Omega, \partial\Omega)$ may have several traces on $\partial\Omega$, in particular the zero function may have the trace 1 on some part of the boundary (see for instance [2]).
- The space $W^{1,2}(\Omega, \partial\Omega)$ is not well suited to deal with inner cracks, even if they are given by smooth hypersurfaces: indeed, due to the density of $W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$, only one trace along the cracks is admitted.
- As $W^{1,2}(\Omega, \partial\Omega)$ may be strictly smaller than $W^{1,2}(\Omega)$ if Ω is not smooth, one may expect that the eigenvalues of the Robin-Laplacian (or the torsion energy) are strictly larger than the ones defined on $W^{1,2}(\Omega)$, provided that their definition in $W^{1,2}(\Omega)$ is possible by some different interpretation of the trace. From this perspective, any Faber-Krahn inequality for arbitrary domains which relies on the Maz'ja space would be weaker than the expected one.

3 Proof of the Saint-Venant inequality for the Robin-Laplacian

The complete proof of Theorem 2.1 with the full comprehension of the non-smooth setting was given in [10] and is based on a free discontinuity approach. In this section, we shall give the main ideas of the proof in the case $q = 1$. This choice brings a little simplification of the exposition. We refer to the full proof and to the general case to reference [10].

Before any technical attack of the problem, let us give the principle of the proof, which can be stated as

$$\text{"Existence and regularity of the optimal set} \Longrightarrow \text{it is the ball"}, \tag{8}$$

from the perspective of the shape optimization problem

$$\min\{E_\beta(\Omega) : \Omega \subseteq \mathbb{R}^N \text{ bounded and open}, |\Omega| = c\}. \tag{9}$$

The proof of this principle relies on reflection arguments which will be described below. We also refer the reader to [8] where a similar approach is employed for the Dirichlet-Laplacian, which naturally raises less technical difficulties. For the minimization of integral functionals, we refer the reader to [20, 19]. This idea is also present in a different setting in [13] and is also related to the celebrated gap in the proof of the isoperimetric inequality by Steiner (see [5]). In fact, Steiner proved that if a smooth two dimensional open set is not the disc, then there exists another set which has the same measure but a strictly lower perimeter. Of course, this does not give a complete proof to the isoperimetric inequality since precisely the existence of a smooth minimizer of the perimeter among sets of prescribed area is missing.

Step 1. Proof of the principle (8)**.** Assume that Ω_{opt} is a solution for (9), which is smooth. Smoothness has to be understood in a weak sense: we require only that the optimal set is open and that the normal can be defined in a certain weak sense, coming from integration by parts. For simplicity of the exposition, let us assume that Ω_{opt} has a piecewise Lipschitz boundary[2] . The optimality of Ω_{opt} entails that Ω_{opt} is connected. Let u be the associated torsion function.

Up to a translation, the torsion function is radially symmetric. Assume that

$$E_\beta(\Omega_{opt}) \leq E_\beta(\Omega),$$

for every piecewise Lipschitz open set Ω. The optimality above can be rewritten as

$$\frac{1}{2}\int_{\Omega_{opt}} |\nabla u|^2 dx + \frac{\beta}{2}\int_{\partial\Omega_{opt}} u^2 dx - \int_{\Omega_{opt}} u dx = E_\beta(\Omega_{opt}) \leq$$

$$\leq E_\beta(\Omega) \leq \frac{1}{2}\int_{\Omega} |\nabla v|^2 dx + \frac{\beta}{2}\int_{\partial\Omega} v^2 dx - \int_{\Omega} v dx,$$

for every function $v \in W^{1,2}(\Omega)$. In order to see that u is radially symmetric, we employ reflection arguments of the following type.

We consider a hyperplane π which divides Ω_{opt} in two parts $\Omega^\pm_{opt}$ with equal volume and denote

$$A^\pm = \frac{1}{2}\int_{\Omega^\pm_{opt}} |\nabla u|^2 dx + \frac{\beta}{2}\int_{\partial\Omega^\pm_{opt}} u^2 dx - \int_{\Omega^\pm_{opt}} u dx.$$

[2]The precise regularity has to be understood in the free discontinuity framework. The piecewise Lipschitz regularity is only required here to support the intuition of the trace.

Employing the algebraic inequality

$$A^+ + A^- \geq 2\min\{A^+, A^-\}, \tag{10}$$

we choose the side with minimal energy and reflect it with respect to π, constructing a new optimal domain, which is now symmetric with respect to π. By symmetrizing successively with respect to hyperplanes parallel to the coordinate axis, we end up with a new domain $\tilde{\Omega}_{opt}$ with a center of symmetry, which we may assume as the new origin of our coordinate system. Note that the fact that we have now a new optimal domain will not have any influence on the uniqueness of a minimizer.

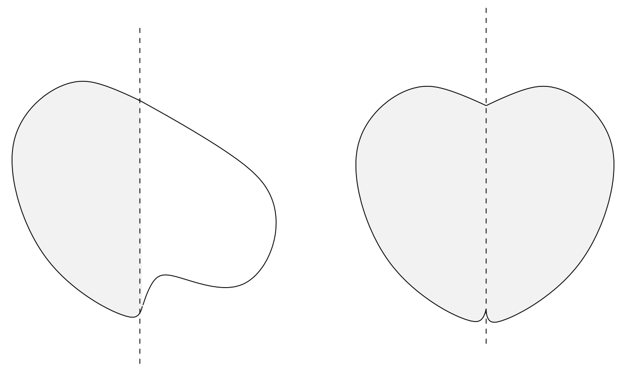


Figure 1: The left side of Ω_{opt} is reflected to build the new optimal domain.

The optimal domain is a ball. The domain $\tilde{\Omega}_{opt}$ is connected thanks to its optimality, with $E_\beta(\tilde{\Omega}_{opt})$ achieved on an function $\tilde{u}$ which is given by successive reflections of the original u associated to Ω_{opt}. Now, every hyperplane π through the origin divides $\tilde{\Omega}_{opt}$ in two parts with equal volume, so that the reflection of at least one of them leads again to a new minimizer of the problem, with associated function given by the reflection of $\tilde{u}$ across π. Since this function should satisfy the associated Euler-Lagrange equation in $\tilde{\Omega}_{opt}$, it is analytic inside $\tilde{\Omega}_{opt}$ and we get that the normal derivative $\frac{\partial \tilde{u}}{\partial n}(x)$ has to vanish on π.

Using analyticity again, and the fact that π was arbitrarily chosen (passing through the origin), we conclude that $\tilde{u}$ is radially symmetric, even though yet it is not clear that $\tilde{\Omega}_{opt}$ is radial. At this point we can go back to the original Ω_{opt}. Since the torsion function u is itself analytic and coincides with $\tilde{u}$ on a portion of Ω_{opt} (which is connected), we have a that u is itself a radial analytic function, and so we have for u a quite restricted choice that can be computed quite explicitly. Indeed we can write $u(x) = \psi(|x|)$ with $\psi : I \to]0, +\infty[$ maximal positive solution of the ordinary differential equation

$$-\psi'' - \frac{N-1}{r}\psi' = 1,$$

which satisfies the *Robin boundary condition*

$$\psi'(|x|) \cdot e_r(x) + \beta\psi(|x|) = 0 \qquad \text{for } x \in \partial\Omega_{opt},$$

where $e_r(x) := x/|x|$. This condition imposes severe restrictions on the shape of Ω_{opt}, entailing for example

$$\partial\Omega_{opt} \subseteq \left\{ x \in \mathbb{R}^N \; : \; \left|\frac{\psi'(|x|)}{\psi(|x|)}\right| \geq \beta \right\}. \tag{11}$$

This represents a great simplification of the problem: the proof that the domain is a ball is nearly straightforward.

Indeed, if $0 \in \overline{\Omega}_{opt}$, one sees that ψ is defined up to the origin with $\psi' \leq 0$ on I and $\psi'(0) = 0$. In view of (11), this entails that $0 \in \Omega_{opt}$, so that Ω_{opt} contains a ball centered at the origin. Denoting by B' the maximal ball contained in Ω_{opt}, one shows (see [10]) that if Ω_{opt} does not coincides with B', then B' is more convenient for E_β: the comparison between $E_\beta(B')$ and $E_\beta(\Omega_{opt})$ is easily exploited by restricting $\psi(|x|)$ on B'. The case $0 \notin \overline{\Omega}_{opt}$ leads with similar arguments to the conclusion that Ω_{opt} coincides with an annulus. Finally, a direct comparison by computation shows that the ball is more convenient than an annulus of the same volume.

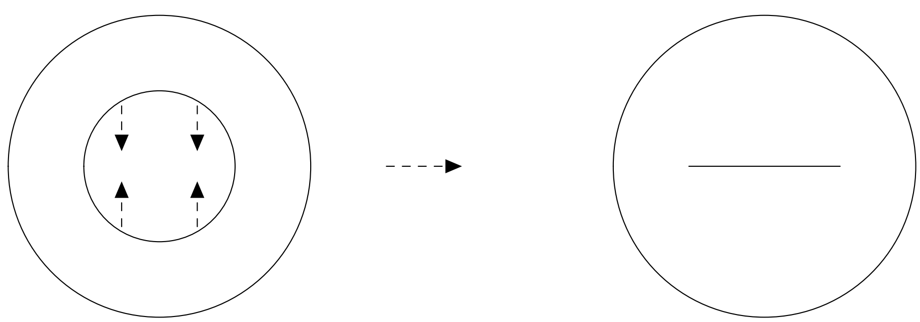
Step 2. Proof of the existence and regularity of Ω_{opt}. This is indeed the difficult and technical part of the proof. In order to prove existence of an optimal domain, one should specify the class of domains where existence holds. Since in general existence is searched in very large classes of shapes, a possibility would be to work in the class of all bounded open sets and with the torsion energy defined through the Maz'ja space:

$$E_\beta^M(\Omega) := \inf_{u \in W^{1,2}(\Omega,\partial\Omega)} \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{\beta}{2}\int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1} - \int_\Omega u dx,$$

so to solve the shape optimization problem

$$\inf\{E_\beta^M(\Omega) : \Omega \subseteq \mathbb{R}^N \text{ bounded and open}, |\Omega| = c\}. \tag{12}$$

This question is well posed, but the strategy to use the Maz'ja spaces for proving existence is not appropriate. This is essentially related to the fact that the Maz'ja space is strictly smaller than the natural space where the problem can be correctly set, for a large class of open sets. We have in mind the example with the cracks which are not seen by the space, but which can naturally occur as a geometric limit of a sequence of smooth sets.



This means that there is no hope to have upper semicontinuity for the torsional rigidity defined through the Maz'ja spaces, for natural geometric variations.

In order to introduce a class of open sets with a relaxed definition of the torsional rigidity which will provide existence, we shift our attention from the domain Ω to the associated torsion function $u_\Omega \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$. Let us view u_Ω as a BV function on $\mathbb{R}^N$ by setting $u_\Omega = 0$ outside Ω. Recall that $v \in BV(\mathbb{R}^N)$ if $v \in L^1(\mathbb{R}^N)$ and the integration by parts formula

$$\forall \varphi \in C_c^\infty(\mathbb{R}^N) \,:\, -\int_{\mathbb{R}^N} v\, div(\varphi)\, dx = \int_{\mathbb{R}^N} \varphi\, dDv$$

holds for a suitable finite measure Dv with values in $\mathbb{R}^N$. Coming back to u_Ω viewed as a BV function on $\mathbb{R}^N$, Du_Ω is composed of a part supported on Ω, absolutely continuous with respect to the volume Lebesgue measure with density ∇u_Ω, and of a part of "jump type" supported on the jump set $J_{u_\Omega} = \partial\Omega$ and absolutely continuous with respect to $\mathcal{H}^{N-1}$. We thus get $u_\Omega \in SBV(\mathbb{R}^N)$, i.e., u_Ω can be interpreted as a *special function of bounded variation.*

The torsion functional can be rewritten as

$$E_\beta(\Omega) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_\Omega|^2\,dx + \frac{\beta}{2}\int_{J_{u_\Omega}} (u_\Omega^+)^2 + (u_\Omega^-)^2\,d\mathcal{H}^{N-1} - \int_{\mathbb{R}^N} u_\Omega dx,$$

where one of the two traces of u_Ω on its jump set is zero. Note that the surface term is rather unusual, involving the *sum* of the traces of u on the jump set. Its form, among the many admissible ones, is chosen in such a way to easily describe the formation of inner cracks by geometric convergence of the domains, and so it is finally dictated by lower semicontinuity requirements.

In view of the preceding equality, we turn our attention to the following free discontinuity functional

$$F_\beta(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{\beta}{2}\int_{J_u} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} - \int_{\mathbb{R}^N} udx \tag{13}$$

and try to minimize it on positive functions $u \in SBV(\mathbb{R}^N)$ with $|\{u > 0\}| \le c$.

The idea is to recover an optimal domain by looking at the support of minimizers. This strategy is subordinated clearly to *regularity issues*, which can ensure that the support is indeed a quite regular domain.

Existence of a minimizer for the free discontinuity problem. The existence of a minimizer u_{opt} for (13) is not straightforward, since the coercivity available for a minimizing sequence is not compatible with Ambrosio's compactness theorem in SBV [1].

By means of the concentration-compactness principle of Pierre-Louis Lions [17, 18], and employing lower semicontinuity properties for the functional F_β, we recover the existence of a *candidate minimizer* $u_{opt} \ge 0$ with $u_{opt}^2 \in SBV(\mathbb{R}^N)$, i.e., such that $F_\beta(u_{opt}) \le F_\beta(v)$ for every admissible $v \in SBV(\mathbb{R}^N)$.

This function can be easily proved to be in L^∞, for instance by the standard De Giorgi technique, but it is not yet in SBV. This last fact is by itself a sort of regularity result, which can be seen as a non degeneracy of the free boundary, related to the Hopf maximum principle. Precisely, one proves the following.

Lemma 3.1 *There exists $\alpha > 0$ such that*

$$\text{for a.e. } x \in \mathbb{R}^N: \quad u_{opt}(x) > 0 \Longrightarrow u_{opt}(x) \ge \alpha.$$

The main consequence of this lemma is that $u_{opt} \in SBV(\mathbb{R}^N)$ and that $\mathcal{H}^{N-1}(J_{u_{opt}})$ is finite.

Ahlfors regularity of the jump set of minimizers. The regularity for u_{opt} we need in order to recover a domain for the torsion problem, reduces to prove that the jump set $J_{u_{opt}}$ is closed. This is a consequence of the fact that the function u_{opt} is an almost quasi-minimizer for the Mumford-Shah functional. Precisely, taking $x \in J_{u_{opt}}$ and comparing the energies for the function u_{opt} and for a function $v \in SBV(\mathbb{R}^N)$ such that $\{u_{opt} \ne v\} \subseteq B_r(x)$ one gets

$$\int_{B_r(x)} |\nabla u_{opt}|^2 dx + \beta\alpha^2\mathcal{H}^{N-1}(J_{u_{opt}} \cap B_r(x)) \\ \le \int_{B_r(x)} |\nabla v|^2 dx + 2\|u_{opt}\|_\infty^2 \beta\mathcal{H}^{N-1}(J_v \cap B_r(x)) + 2r^N(\omega_N + d)\|u_{opt}\|_\infty,$$

for some value $d > 0$, provided that the radius r is smaller than some constant.

Following the general Ahlfors regularity result of [12], the quasi-minimality property of u_{opt} implies that $J_{u_{opt}}$ is essentially closed ($\mathcal{H}^{N-1}(\overline{J_{u_{opt}}} \setminus J_{u_{opt}}) = 0$) and that it satisfies for $\mathcal{H}^{N-1}$-a.e. point $x \in J_{u_{opt}}$ and for small radius r

$$C \leq \frac{\mathcal{H}^{N-1}(J_{u_{opt}} \cap B_r(x))}{r^{N-1}} \leq \frac{1}{C},$$

where $C > 0$ is a suitable constant.

Let Ω_{opt} be the union of the connected components of $\mathbb{R}^N \setminus \overline{J}_{u_{opt}}$ where u_{opt} is not vanishing. This is an open set whose boundary is contained in $\overline{J}_{u_{opt}}$, which is a rectifiable set with finite $\mathcal{H}^{N-1}$-measure. By optimality we get that Ω_{opt} is connected and that $|\Omega_{opt}| = c$.

The class $\mathcal{A}(\mathbb{R}^N)$ of admissible domains. In view of the regularity properties of the minimizers of (13) described above, we are led to introduce the following class of domains

$$\mathcal{A}(\mathbb{R}^N) := \{\Omega \subseteq \mathbb{R}^N \,:\, \Omega \text{ is open with } |\Omega| < +\infty, \\ \text{and } \partial\Omega \text{ is rectifiable with } \mathcal{H}^{N-1}(\partial\Omega) < +\infty\}. \quad (14)$$

The *rectifiability* of $\partial\Omega$ is a sort of piecewise regularity in the sense of geometric measure theory: it means that $\partial\Omega$ is contained, up to $\mathcal{H}^{N-1}$-negligible sets, into the union of a countable family of C^1-regular manifolds. This weak regularity requirement is readily seen to be stable under intersections and reflections. Moreover a normal vector field ν on $\partial\Omega$ can be defined, and this is an important information for the Robin condition. Finally domains in $\mathcal{A}(\mathbb{R}^N)$ have finite perimeter, so that a weak form of the integration by parts is still available.

In view of the form of the free discontinuity functional, for every $\Omega \in \mathcal{A}(\mathbb{R}^N)$ we define

$$E_\beta(\Omega) := \inf_{u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)} \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\beta}{2}\int_{\partial\Omega} (u^+)^2 + (u^-)^2\,d\mathcal{H}^{N-1} - \int_\Omega u dx. \quad (15)$$

The traces in (15) are well defined since, after extending by zero outside Ω, u belongs to $SBV(\mathbb{R}^N)$. In view of the fine properties of BV functions, at $\mathcal{H}^{N-1}$-a.e. point of $x \in \partial\Omega$ with normal $\nu(x)$, the two values

$$u^\pm(x) = \lim_{r\to 0^+} \frac{1}{|B_r^\pm(x,\nu(x))|}\int_{B_r^\pm(x,\nu(x))\cap\Omega} u(y)\,dy$$

are well defined, where $B_r^\pm(x,\nu(x)) := \{y \in B_r(x) \,:\, (y-x)\cdot\nu(x) \gtrless 0\}$, and the integrals in (15) can be computed.

Note that the form of $E_\beta(\Omega)$ takes automatically into account that $\partial\Omega$ may contain "cracks", so that the two traces $u^\pm$ on both sides have to be considered. Naturally, the trace coming from outside Ω, i.e. from Ω^c, is vanishing, so that $u^+ = 0$. Finally, if Ω was Lipschitz, the torsion energy clearly reduces to the classical one.

The analysis of the free discontinuity functional (13) entails that for a minimizer u_{opt} we have that the associated support Ω_{opt} belongs to $\mathcal{A}(\mathbb{R}^N)$ and that

$$F_\beta(u_{opt}) = E_\beta(\Omega_{opt}).$$

This easily entails that the minimization of E_β on $\mathcal{A}(\mathbb{R}^N)$ under a volume constraint is well posed.

Theorem 3.2 *For every $\beta, c > 0$ the problem*

$$\min\{E_\beta(\Omega) : \Omega \in \mathcal{A}(\mathbb{R}^N), |\Omega| = c\}, \tag{16}$$

has a solution.

Theorem 3.2 together with Principle (8) entail the validity of Theorem 1.1, since the torsion energy reduces to the classical one for Lipschitz regular sets. Indeed, we recover a strengthened optimality property for the ball, since also nonsmooth domains are taken into account.

Finally, as detailed in [9, Section 3], if $\Omega \subseteq \mathbb{R}^N$ is open and bounded, and $u \geq 0$ belongs to the Maz'ja space $W^{1,2}(\Omega, \partial\Omega)$, then after an extension by zero outside the domain, we have $u^2 \in SBV(\mathbb{R}^N)$ with

$$F_\beta(u) \leq \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{\beta}{2}\int_{\partial\Omega} u^2\, d\mathcal{H}^{N-1} - \int_\Omega u dx.$$

This inequality easily implies that the solution of problem (12) is still the ball.

Remark 3.3 The class $\mathcal{A}(\mathbb{R}^N)$ defined in (14) provides a natural framework for the shape optimization problems under Robin boundary conditions: in this direction we refer the reader to [11].

4 Open questions

Below we present two open questions which occur naturally in the context of Faber-Krahn inequalities for the Robin-Laplacian.

Open problem 1.

A. Let $p \in (1, +\infty]$ and $\Omega \subseteq \mathbb{R}^N$ a bounded Lipschitz set. Is it true that among all Lipschitz domains of prescribed measure, the one which maximizes the p-norm of the torsion function for the Robin-Laplacian is the ball? In other words, is it true that $|u_\Omega|_p \leq |u_{\Omega^*}|_p$ for $p \in (1, +\infty]$?

B. A stronger version of the question above is the following. Is it true that

$$u^*_\Omega(x) \leq u_{\Omega^*}(x), \text{ a.e. } x \in \Omega^*?$$

In the case of Dirichlet boundary conditions, both problems have a positive answer given by Talenti's rearrangement theorem. In this paper, we gave a positive answer to Problem A in the case $p = 1$. For the other values of p, our proof does not work.

Open problem 2. The eigenvalue problem (5) can be correctly set for every $q \in [1, \frac{2N}{N-1})$. Nevertheless, the validity of the Faber-Krahn inequality stating that

$$\lambda^\beta_{1,q}(\Omega) \geq \lambda^\beta_{1,q}(\Omega^*)$$

was proved only for the case $q \in [1, 2]$ (see [10]). For $q \in (2, \frac{2N}{N-1})$ only a (weaker) penalized version is proved to hold, namely that for every $c > 0$

$$\lambda^\beta_{1,q}(\Omega) + c|\Omega| \geq \lambda^\beta_{1,q}(\Omega^*) + c|\Omega^*|.$$

In order to prove the strong version for $q \in (2, \frac{2N}{N-1})$ a way to attack this problem is to rely on the weak version and to prove that the mapping

$$(0, +\infty) \ni r \mapsto \lambda_{1,q}^{\beta}(B_r)$$

is convex.

In appearance this is a simple problem, since it deals only with the behavior of the eigenvalue on balls. Nevertheless, some technical difficulties have to be faced and a smart idea has to be found in order to prove convexity for $q \in (2, \frac{2N}{N-1})$.

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